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Covariant SPDEs and quantum field structures

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Abstract. Covariant stochastic partial differential equations (SPDEs) are studied in any dimension. A special class of such equations is selected and it is proved that the solutions can be analytically continued to Minkowski spacetime yielding tempered Wightman distributions which are covariant, obey the locality axiom and a weak form of the spectral axiom.

1. Introduction

The connection between scalar generalized random fields which are Markov and Euclidean invariant and scalar quantum fields played a crucial role in the development of constructive quantum-field theory [24, 43]. Symanzik [48] first pointed out this connection for the free field and Nelson [36, 37] developed some general machinery to construct quantum fields from Euclidean invariant Markov fields. Multicomponent Gaussian generalized random fields which are Markov and invariant under the Euclidean group might play a role similar to that of the free scalar field [26, 50–52]. A simple example for such covariant random fields is given by infinitely divisible random fields [10]. It seems that these fields are too singular: perturbations by local multiplicative functionals as in the standard constructive quantum-field-theory approach should lead to very serious ultraviolet divergence problems; nevertheless there is another constructive approach which was initiated in [1–4] and in the following papers [5, 6, 38, 39]. In all the above-mentioned papers dealing with vector fields it is essential that a real vector space of dimension $D = 1, 2, 4, 8$ can be given the structure of a division algebra so that the Laplace operator $\triangle_D = \sum_{i=1}^{D} \frac{\partial^2}{\partial x_i^2}$ can be factorized as a product of two first-order covariant elliptic differential operators ∂ and $\bar{\partial}$. One can then consider an equation of the form

$$\partial A = \eta \tag{1}$$

where η is suitably chosen noise. The solution of this equation, which can be computed explicitly, is again a covariant Markovian generalized random field. The moments of this generalized random field can be analytically continued to Minkowski spacetime, yielding a covariant system of Wightman distributions which obey the locality axiom and a weak form of the spectral axiom [12, 28, 44]. By a weak form of the spectral axiom we mean here that the Fourier transforms of the corresponding Wightman distributions are supported in products of the closed forward lightcones. Moreover, if the noise η contains a nonzero Poisson piece the corresponding system of Wightman functions is not quasi-free (non-Gaussian).

In this paper we shall consider an equation of the type

$$\mathcal{D}A = \eta \tag{2}$$

in arbitrary spacetime dimension $D \geqslant 2$ and where $\mathcal{D}$ is an arbitrary covariant differential operator of any order.

It is among the main objectives of this paper to demonstrate that the existence of division algebras in the particular dimensions $D = 1, 2, 4$, and 8 is not essential and that in any dimension a covariant Markovian generalized random field A can be constructed by solving equation (2) with suitable $\mathcal{D}$ and η. Moreover it will be shown that it is a generic property of a large class of such equations that the moments of the random field A can be analytically continued to Minkowski space giving a set of tempered Wightman distributions which are covariant and which fulfil the locality axiom and a weak form of the spectral axiom.

The essential problem behind these constructions is to decide whether a reflection-positive non-Gaussian covariant generalized random field A can be obtained from equation (2). Unfortunately, the authors have obtained some partial negative results which will be published in forthcoming papers [8, 9, 20]. For a construction of Gaussian Euclidean fields of arbitrary spin in an axiomatic framework we refer to [42].

It seems to be an intrinsic property of gauge fields that the conditions of positivity, covariance and locality are all together not compatible with local gauge invariance [45, 46]. In view of this, we expect that some of the models produced by the methods described in this paper, though they are not reflection-positive, could find applications in problems of quantum-field theory of gauge type with indefinite metrics. This is the second motivation for this and some forthcoming papers [19, 34]. Examples of Gaussian reflection-positive covariant random fields are contained in [52, 53].

1.1. Organization of the paper

Although the proper mathematical language for the material presented in this paper is the language of vector bundles over $\mathbb{R}^D$ and equivariant differential operators of first order [31, 7, 21] we decided to present our results in a more elementary way in order to make them easily accessible. In section 2 we fix the notation and mention some elementary results which some of the readers probably know. The main result of the paper is contained in section 3: assume that $\mathcal{D}$ has an admissible mass spectrum (see below for the definition) and that η is white noise that possesses all moments. Then there exist tempered covariant distributions supported in the forward cone such that their Fourier–Laplace transforms are equal to the moments of A regarded as functions of the difference variables at positive time. Finally, in the last section we present some particular examples in three-dimensional space resulting from the lowest-dimensional real representations of the group $\boldsymbol{SO}(\mathbf{3})$. Models describing the interaction between scalar fields and vector fields that we call Higgs$_3$-like models and models describing two interacting vector fields are also presented in the last section.

2. Random fields as solutions of covariant SPDEs

2.1. Covariant first-order differential operators

An important concept in physics is the concept of covariance, i.e. the fact that the form of an equation does not change under suitable coordinate transformations. There is a lot of literature on this subject [15, 17, 25, 32, 49]. In this section we shall investigate covariant first-order differential operators acting on C^∞-functions $\mathbb{R}^D \to \mathbb{K}^N$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We assume that a representation of a Lie group $\boldsymbol{G} \subseteq \boldsymbol{GL(D)}$ is acting on $\mathbb{K}^N$. In our applications we shall mainly study the case $\boldsymbol{G} = \boldsymbol{SO(D)}$, which is motivated by our intention to produce covariant models in the framework of Euclidean quantum-field theory.

Let us, first of all, collect some basic definitions and facts.

Proposition 2.1. Let τ be a representation of the Lie group $\boldsymbol{G}$ in Aut$\mathbb{K}^N$. Let $B_1, \dots, B_D$ be matrices $\in \mathcal{M}_{N\times N}(\mathbb{K})$ and put $B = (B_1, \dots, B_D)$. Let $E \in \mathcal{M}_{N\times N}(\mathbb{K})$ denote the unit matrix.

We consider the first-order operator

$$\mathcal{D}_B = \sum_{j=1}^{D} B_j \frac{\partial}{\partial x_j} + mE \qquad m \in \mathbb{R} \tag{3}$$

acting on the space of C^∞-functions $\mathbb{R}^D \to \mathbb{K}^N$. Let T_g^τ denote the action of the representation τ on functions $f \in C^\infty(\mathbb{R}^D, \mathbb{K}^N)$:

$$T_g^\tau f(x) = \tau(g) f(g^{-1}x) \qquad g \in \boldsymbol{G}. \tag{4}$$

The following statements are equivalent.

(a) The form of $\mathcal{D}_B$ does not change if we make a coordinate transformation in $\mathbb{R}^D : x \mapsto gx,\ g \in \boldsymbol{G}$, and simultaneously a coordinate transformation in $\mathbb{K}^N$: $y \mapsto \tau(g)y$.

(b) $\mathcal{D}_B$ commutes with T_g^τ:

$$[\mathcal{D}_B, T_g^\tau] = 0 \qquad \forall g \in \boldsymbol{G}. \tag{5}$$

(c)

$$\sum_{k=1}^{D} g_{jk} \tau(g) B_k \tau(g^{-1}) = B_j \qquad \forall j \in \{1, \dots, D\} \qquad \forall g \in \boldsymbol{G} \tag{6}$$

where g_{jk} are the components of $g \in \boldsymbol{G}$.

Note that instead of taking the operator $m \cdot E$ in (3) we can take any matrix M belonging to the centre of the image of τ.

Definition 2.2. If $\mathcal{D}_B$ fulfils one (and hence all) of the conditions in proposition 2.1, it will be called covariant with respect to the representation τ.

The set of all operators that are covariant with respect to τ will be denoted by Cov$(\mathbb{K}^N, \tau)$.

Note that if $\tau(g) \in \boldsymbol{O(N)}\ \forall g \in \boldsymbol{G}$ and if $(B_1, \dots, B_D)$ defines a covariant operator with respect to τ then the transposed matrices $(B_1^t, \dots, B_D^t)$ define a covariant operator with respect to τ, too.

If we omit the constant term in equation (3), we can be a little more general: In this situation we can also admit matrices B_j that are not quadratic, i.e. we can consider operators $\mathcal{D}_B : C^\infty(\mathbb{R}^D, \mathbb{K}^N) \to C^\infty(\mathbb{R}^D, \mathbb{K}^M)$.

Proposition 2.3. Let τ be a representation of the group $\boldsymbol{G}$ in Aut$\mathbb{K}^N$ and let σ be a representation of $\boldsymbol{G}$ in Aut$\mathbb{K}^M$. Let $B_1, \dots, B_D \in \mathcal{M}_{M\times N}$ and put $B = (B_1, \dots, B_D)$.

We consider the operator $\mathcal{D}_B$ defined in equation (3) and put $m = 0$. Let T_g^τ denote the action of τ in $C^\infty(\mathbb{R}^D, \mathbb{K}^N)$ and let S_g^σ denote the action of σ in $C^\infty(\mathbb{R}^D, \mathbb{K}^M)$. The following statements are equivalent.

(a) The form of $\mathcal{D}_B$ does not change if we make a coordinate transformation in $\mathbb{R}^D : x \mapsto gx, g \in \boldsymbol{G}$, and simultaneously coordinate transformations in $\mathbb{K}^N : y \mapsto \tau(g)y$ and in $\mathbb{K}^M : z \mapsto \sigma(g)z$.

(b) $\mathcal{D}_B$ intertwines T_g and S_g:

$$\begin{array}{ccc} C^\infty(\mathbb{R}^D, \mathbb{K}^N) & \xrightarrow{\mathcal{D}_B} & C^\infty(\mathbb{R}^D, \mathbb{K}^M) \\ T_g^\tau \big\downarrow & & \big\downarrow S_g^\sigma \\ C^\infty(\mathbb{R}^D, \mathbb{K}^N) & \xrightarrow{\mathcal{D}_B} & C^\infty(\mathbb{R}^D, \mathbb{K}^M). \end{array}$$

(c)

$$\sum_{k=1}^{D} g_{jk}\sigma(g)B_k\tau(g^{-1}) = B_j \qquad \forall j \in \{1, \ldots, D\} \qquad \forall g \in \boldsymbol{G}$$

where g_{jk} are the components of $g \in \boldsymbol{G}$.

For given τ and σ the set of all operators fulfilling one of the conditions of proposition 2.3 will be denoted as $\mathrm{Cov}((\tau, \mathbb{K}^N); (\sigma, \mathbb{K}^M))$. The following lemma is the infinitesimal version of the transformation properties (6).

Lemma 2.1. Let $\boldsymbol{g}$ denote the Lie algebra of $\boldsymbol{G}$, and let $L_\alpha, \alpha \in \{1, \ldots, l\}$, be a family of generators of $\boldsymbol{g}$.

A necessary condition that a D-tuple of matrices $B = (B_1, \ldots, B_D)$ defines a covariant operator $\mathcal{D}_B$ with respect to the representation τ is that

$$\sum_{k=1}^{D} (L_\alpha)_{jk} B_k = [B_j, \mathrm{d}\tau(L_\alpha)] \qquad \forall \alpha \in \{1, \ldots, l\} \qquad \forall j \in \{1, \ldots, D\} \tag{7}$$

where $\mathrm{d}\tau$ denotes the differential of τ.

If $\boldsymbol{G}$ is connected, condition (7) is also sufficient.

Sketch of the proof. The infinitesimal form follows easily from the global condition. Therefore we shall concentrate on the proof of the inverse implication. First we show that the statement to be proved holds for one-parameter groups. Let us take the one-parameter group $g(t) = \mathrm{e}^{\mathrm{i}tL_\alpha}$ and its representation $T_g(t) = \mathrm{e}^{\mathrm{i}t\,\mathrm{d}\tau(L_\alpha)}$. By the commutator expansion we have

$$\sum_k T_g B_k T_g^{-1} g_{ik} = \sum_k \sum_{n\geqslant 0} \frac{\mathrm{i}^n t^n}{n!} [\mathrm{d}\tau(L_\alpha), \ldots, [\mathrm{d}\tau(L_\alpha), B_k] \ldots](\mathrm{e}^{\mathrm{i}tL_\alpha})_{ik}.$$

Iterating (5) we obtain

$$\begin{aligned} [\mathrm{d}\tau(L_\alpha), \ldots, [\mathrm{d}\tau(L_\alpha), B_k] \ldots] &= -\left[\mathrm{d}\tau(L_\alpha), \ldots, \left[\mathrm{d}\tau(L_\alpha), \sum_{k_1} B_{k_1}(L_\alpha)_{kk_1}\right] \ldots\right] \\ &= -\sum_{k_1} (L_\alpha)_{kk_1} [\mathrm{d}\tau(L_\alpha), \ldots, [\mathrm{d}\tau(L_\alpha), B_{k1}] \ldots] \\ &= (-1)^n \sum_{k_1 \ldots k_n} (L_\alpha)_{kk_1}(L_\alpha)_{k_1k_2} \ldots (L_\alpha)_{k_{n-1}k_n} B_{k_n} \end{aligned}$$

so that

$$\begin{aligned} \sum_k T_g B_k T_g^{-1} g_{ki} &= \sum_{n\geqslant 0} \frac{(-i)^n t^n}{n!} \sum_k \sum_{k_1 \ldots k_n} (-1)^n (\mathrm{e}^{\mathrm{i}tL_\alpha})_{ik} (L_\alpha)_{kk_1}(L_\alpha)_{k_1k_2} \ldots (L_\alpha)_{k_{n-1}k_n} B_{k_n} \\ &= \sum_{n\geqslant 0} \frac{(-i)^n t^n}{n!} \sum_{k_n} (\mathrm{e}^{\mathrm{i}tL_\alpha} L_\alpha^n)_{ik_n} B_{k_n} = \sum_k \left[\mathrm{e}^{\mathrm{i}tL_\alpha} \left(\sum_{n\geqslant 0} \frac{(-i)^n t^n}{n!} L_\alpha^n\right)\right]_{ik} B_{k_n} = B_i. \end{aligned}$$

This completes the proof for one-parameter groups. Since

$$\sum_k T_{g_2g_1} B_k T^{-1}_{g_2g_1} (g_2g_1)_{lk} = \sum_k \sum_i T_{g_2} T_{g_1} B_k T^{-1}_{g_1} T^{-1}_{g_2} g_{1,ik} g_{2,li} = \sum_i T_{g_2} B_i T_{g_2} g_{2,li} = B_l$$

and the fact that the statement holds for one-parameter groups means we have proved the implication for group elements g which are products $g = g_1(t_1) \dots g_k(t_k)$ of elements from the one-parameter groups $g_i(t_i)$. The set of such products is dense in some open subset U containing the identity. By the continuity argument the global condition is fulfiled on U, and consequently is fulfiled on the connected component containing the identity. □

Remark 2.2. Let the Lie group $\boldsymbol{G}$ be the union of connected components $\boldsymbol{G} = \bigcup_\alpha \boldsymbol{G}^\alpha$ with $\boldsymbol{G}^0$ being the connected component containing the unit element $\boldsymbol{e}$. Assume that there exist(s) $R_\alpha \in \boldsymbol{G}$ such that $R_\alpha \boldsymbol{G}^0 = \boldsymbol{G}^\alpha$. If for a given representation τ equations (7) hold and if

$$\sum_{k=1}^{D} (R_\alpha)_{jk} \tau(R_\alpha) B_k \tau^{-1}(R_\alpha) = B_j \tag{8}$$

then the D-tuple $(B_j)_{j=1,\dots,D}$ defines a covariant operator $\mathcal{D}$ under the action of the component(s) $\boldsymbol{G}^\alpha$.

Similarly we can also prove the following lemma.

Lemma 2.3. Let $\boldsymbol{G}$, $\boldsymbol{g}$, L_α be as in lemma 2.1 and let σ, τ be two representations of $\boldsymbol{G}$ in $\mathrm{Aut}\mathbb{K}^N$ and in $\mathrm{Aut}\mathbb{K}^M$ respectively. A necessary condition that a D-tuple of matrices $B = (B_1, \dots, B_D)$ defines a covariant operator $\mathcal{D}_B \in \mathrm{Cov}((\tau, \mathbb{K}^M), (\sigma, \mathbb{K}^N))$ is that

$$\sum_{k=1}^{D} (L_\alpha)_{jk} B_k + \mathrm{d}\sigma(L_\alpha) B_j + B_j \, \mathrm{d}\tau(L_\alpha) = 0 \tag{9}$$

for all $j \in \{1, \dots, D\}$ and $\alpha = 1, \dots, \dim \boldsymbol{G}$.

If $\boldsymbol{G}$ is connected this condition is also sufficient.

For the case of the rotation group $\boldsymbol{SO(3)}$ in three-dimensional space and also for the proper orthochronous Lorentz group $\boldsymbol{L}^\uparrow_+\boldsymbol{(4)}$ in four-dimensional spacetime covariant operators have been extensively studied, see [17, 32, 49] and the references therein.

In the following we wish to study the inverse of a given covariant operator. It is therefore natural to ask whether we can find any elliptic operators in $\mathrm{Cov}(\mathbb{K}^N, \tau)$.

For an operator $\mathcal{D}_B = \sum_{j=1}^D B_j \frac{\partial}{\partial x_j} + mE$ and a differential form $\sum_{j=1}^D p_j \mathrm{d}x_j$ we define the characteristic polynomial in the usual way:

$$\sigma_{\mathcal{D}_B}(p_1, \dots, p_D) \overset{\text{def}}{=} \mathrm{i} \sum_{j=1}^{D} B_j p_j .$$

Note that this definition depends in general on the choice of coordinates.

Lemma 2.4. (a) Let $\boldsymbol{G} \subseteq \boldsymbol{O(D)}$ and let $\mathcal{D}_B \in \mathrm{Cov}(\mathbb{K}^N, \tau)$.

The form of $\sigma_{\mathcal{D}_B}(p_1, \dots, p_D)$ does not change if we make a coordinate transformation in $\mathbb{R}^D$: $x \mapsto gx$, $g \in \boldsymbol{G}$, and simultaneously a coordinate transformation in $\mathbb{K}^N$: $y \mapsto \tau(g)y$.

(b) Let $\boldsymbol{G}$ be either $\boldsymbol{SO(D)}$ or $\boldsymbol{O(D)}$ and let $\mathcal{D}_B \in \mathrm{Cov}(\mathbb{K}^N, \tau)$.

We have

$$\det(\sigma_{\mathcal{D}_B}(p_1, \dots, p_D)) = C(p_1^2 + \dots + p_D^2)^n \tag{10}$$

for some constant $C \in \mathbb{C}$, $n \in \boldsymbol{N}$ and such that $n \leqslant N/2$.

Moreover, if N is odd $\det(\sigma_{\mathcal{D}_B}(p_1,\ldots,p_D)) = 0$, i.e. elliptic operators that are covariant with respect to some representation of $\boldsymbol{SO(D)}$ or $\boldsymbol{O(D)}$ can only exist if the dimension of the representation space is even.

Proof. (a) is easily seen by employing the covariance condition (6). To prove (b), observe that $\det(\sigma_{\mathcal{D}_B}(p_1,\ldots,p_D))$ is invariant under rotations and must therefore be a function of $p_1^2+\cdots+p_D^2$. The assertion now follows from the fact that $\det(\sigma_{\mathcal{D}_B}(p_1,\ldots,p_D))$ must be a polynomial in the p_j's homogeneous of order less than or equal to N. □

Remark 2.5. Let $\boldsymbol{G}$ be either $\boldsymbol{SO(D)}$ or $\boldsymbol{O(D)}$ and let N be even. For a covariant operator $\mathcal{D}_B \in \mathrm{Cov}(\mathbb{K}^N, \tau)$ we have

$$\det\left(\mathrm{i}\sum_{j=1}^{D} B_j p_j + mE\right) = C\prod_{\alpha=1}^{n}(p_1^2+\cdots+p_D^2+m_\alpha^2) \tag{11}$$

where $m_\alpha, \alpha = 1,\ldots,n, n \leqslant \frac{N}{2}$, and C are constants $\in \mathbb{C}$.

If all m_α are real and $C \neq 0$, the operator $\mathcal{D}_B$ is invertible on suitably chosen function spaces and in this case we shall call it admissible. If all $m_\alpha \neq 0$, operator $\mathcal{D}_B$ is said to have a strictly positive mass spectrum.

Given two different but equivalent representations τ and $\tilde{\tau}$, the following remark shows how we can identify $\mathrm{Cov}(\mathbb{K}^N, \tau)$ and $\mathrm{Cov}(\mathbb{K}^N, \tilde{\tau})$.

Remark 2.6. We assume that $B = (B_1,\ldots,B_D)$ defines a covariant operator with respect to the representation τ. Let $\tilde{\tau}$ be an equivalent representation: $\tilde{\tau}(g) = M\tau(g)M^{-1}$.

Then $B' = (B'_1,\ldots,B'_D)$, $B'_j = MB_jM^{-1}$, defines a covariant operator with respect to $\tilde{\tau}$.

Remark 2.7. It is possible to consider covariant differential operators of higher order, too. For this let

$$\mathcal{D}_n = \sum_{|\alpha|\leqslant n} B_\alpha \partial_\alpha + M$$

where $\alpha = (\alpha_1,\ldots,\alpha_D)$, $\alpha_i \in \mathbb{N}\cup\{0\}$, $|\alpha| = \alpha_1+\cdots+\alpha_D$, $B_\alpha \in \mathcal{M}_{N\times N}(\mathbb{K})$, $\partial_\alpha = \frac{\partial^{\alpha_1+\ldots\alpha_D}}{\partial x_D^{\alpha_D}\ldots\partial x_1^{\alpha_1}}$ and let τ be a representation of the group $\boldsymbol{G}$ in $\mathrm{Aut}\mathbb{K}^N$. Then the operator $\mathcal{D}_n$ is called a τ-covariant differential operator of order n iff

(i) there exists α such that $|\alpha| = n$ and $B_\alpha \neq 0$,

(ii) the following diagrams commute:

$$\begin{array}{ccc} C^\infty(\mathbb{R}^D,\mathbb{K}^N) & \xrightarrow{\mathcal{D}_n} & C^\infty(\mathbb{R}^D,\mathbb{K}^N) \\ T_g^\tau\big\downarrow & & \big\downarrow T_g^\tau \\ C^\infty(\mathbb{R}^D,\mathbb{K}^N) & \xrightarrow{\mathcal{D}_B} & C^\infty(\mathbb{R}^D,\mathbb{K}^N). \end{array}$$

In particular, taking $\mathcal{D}^1,\ldots,\mathcal{D}^n \in \mathrm{Cov}(\tau;\mathbb{K}^N)$, the operator $\mathcal{D}_n = \mathcal{D}^n\ldots\mathcal{D}^1$ is a covariant operator of nth order. However, since by increasing the dimension N of the target space $\mathbb{K}^N$ the nth order covariant equation can be reduced to first order, we shall mainly restrict ourselves to first-order operators.

Let us now focus on the case $\boldsymbol{G} = \boldsymbol{SO(D)}$. The representation theory of $\boldsymbol{SO(D)}$ is well known [11, 14, 16, 17]. An important question for physics is which representations τ of $\boldsymbol{G} = \boldsymbol{SO(D)}$ admit an extension to a representation $\tilde{\tau}$ of $\boldsymbol{O(D)}$. Since $\boldsymbol{SO(D)}$ is a subgroup of index 2 of $\boldsymbol{O(D)}$, it is a normal subgroup and $\boldsymbol{O(D)}/\boldsymbol{SO(D)} \cong \mathbb{Z}_2$. Taking any $M \in \boldsymbol{O(D)} \setminus \boldsymbol{SO(D)}$ it is easy to check that τ can be extended to $\boldsymbol{O(D)}$ iff there exists $\tilde{\tau}(M) \in \mathcal{M}_{N\times N}(\mathbb{K})$ such that

$$\tau(MAM) = \tilde{\tau}(M)\tau(A)\tilde{\tau}(M) \qquad \forall A \in \boldsymbol{SO(D)}. \tag{12}$$

If D is odd one can always extend a given representation τ: the fact that D is odd implies that the matrix $M = -E_D = (-\delta_{ij})$ has determinant -1, and if we put $\tilde{\tau}(M) = \pm\mathrm{id}_V$, condition (12) is fulfilled.

Let us now have a look at

$$R = \begin{pmatrix} -1 & 0 \\ 0 & E_{D-1} \end{pmatrix} \tag{13}$$

which is the reflection at the hyperplane $\{x_1 = 0\}$. The choice $\tilde{\tau}(-E_D) = \pm\mathrm{id}_V$ implies that the reflection R is represented by

$$\tilde{\tau}(R) = \pm\tau \begin{pmatrix} 1 & 0 \\ 0 & -E_{D-1} \end{pmatrix}. \tag{14}$$

The case of even dimension is more complicated so that we only give a summary of some group-theoretic results, referring the reader to [11] for details.

We assume that τ is an irreducible unitary representation. Taking some $M \in \boldsymbol{O(D)} \setminus \boldsymbol{SO(D)}$, we consider the representation $\sigma(A) = \tau(M^{-1}AM)$, $A \in \boldsymbol{SO(D)}$. If σ and τ are equivalent, τ is called self-conjugate. In this case τ can be extended to $\boldsymbol{O(D)}$, and the extension is unique up to sign. If, however, σ and τ are not equivalent, one has to pass to the induced representation τ_{ind} of $\boldsymbol{O(D)}$, i.e. one has to double the dimension of the representation space $\mathbb{K}^N$. τ_{ind} is an irreducible representation of $\boldsymbol{O(D)}$, and it is the only irreducible representation of $\boldsymbol{O(D)}$ which contains τ when being restricted to $\boldsymbol{SO(D)}$.

Now we can introduce reflections into the concept of covariant operators.

Definition 2.4. Let τ be a representation of $\boldsymbol{G} = \boldsymbol{SO(D)}$, and let $\tilde{\tau}$ be an extension of τ to $\boldsymbol{O(D)}$.

We call an operator $\mathcal{D}_B \in \mathrm{Cov}(\mathbb{K}^N, \tau)$ reflection covariant with respect to $\tilde{\tau}$ iff it transforms covariantly under the full orthogonal group, i.e. if (6) holds $\forall g \in \boldsymbol{O(D)}$.

Remark 2.8. Let $\mathcal{D}_B$ be a covariant operator with respect to a representation τ of $\boldsymbol{SO(D)}$, and let $\tilde{\tau}$ be an extension of τ to $\boldsymbol{O(D)}$. $\mathcal{D}_B$ is reflection covariant with respect to $\tilde{\tau}$ iff

$$\begin{aligned} &\tilde{\tau}(R)B_1\tilde{\tau}(R) = -B_1 \\ &\tilde{\tau}(R)B_j\tilde{\tau}(R) = B_j \qquad \forall j \in \{2, \dots, D\} \end{aligned} \tag{15}$$

where R is the matrix in equation (13).

Unitary representations of the classical groups are well understood. In the following we use representations in terms of real matrices.

Let V be a complex finite-dimensional vector space. Given a representation $\tau : \boldsymbol{G} \to \mathrm{Aut}V$, it is natural to ask whether τ can somehow be transformed into a representation in terms of real matrices. A comprehensive treatment of this question can be found in [14, 16].

τ is *of real type* iff there is an antilinear map $J : V \to V$ such that $J^2 = \mathrm{id}_V$ and $J\tau(g) = \tau(g)J \forall g \in \boldsymbol{G}$.

If τ is of real type, consider $W = \{x \in V | x = Jx\}$. W is a real subspace which is $\tau(g)$-invariant $\forall g \in \boldsymbol{G}$. We have the decomposition $V = W \oplus \mathrm{i}W$ which shows that τ can be obtained from $\tau_{\mathrm{real}} : \boldsymbol{G} \to W$ by extending the field of scalars. Choosing a basis for W, we get a representation in terms of real matrices.

τ is *of quaternionic type* iff there is an antilinear map $J : V \to V$ such that $J^2 = -\mathrm{id}_V$ and $J\tau(g) = \tau(g)J \forall g \in \boldsymbol{G}$. If the representation τ is of quaternionic type, it can be extended to $\tau_{\mathrm{quat}} : V \oplus jV$, where $\{1, i, j, k\}$ denotes, as usual, the canonical basis for the space of quaternions.

If τ is neither of real nor of quaternionic type, we say that τ is *of complex type*. The following proposition is a well known criterion to determine the type of a given irreducible representation.

Proposition 2.5. Let $\mathrm{d}g$ denote the normalized Haar measure on the compact Lie group $\boldsymbol{G}$ and let χ_τ denote the character of the irreducible representation $\tau : \boldsymbol{G} \to \mathrm{End}V$.

$$\int_{\boldsymbol{G}} \chi_\tau(g^2)\,\mathrm{d}g = \begin{cases} 1 & \iff \tau \textit{ is of real type} \\ 0 & \iff \tau \textit{ is of complex type} \\ -1 & \iff \tau \textit{ is of quaternionic type.} \end{cases}$$

For the proof see e.g. [54].

2.2. Non-Gaussian noise

In this section we shall deal with $\boldsymbol{G}$-invariant and reflection-positive noise. Since mathematical physicists might be less acquainted with the notion of non-Gaussian noise, we briefly review some basic definitions and facts.

Definition 2.6. Let (Ω, Σ, μ) be a probability space, and let T be a space of smooth test functions $\mathbb{R}^D \to \mathbb{R}^N$. We assume that T is equipped with some topology.

A generalized random field indexed by T is a map

$$\varphi : T \longrightarrow \{\text{real-valued random variables on } \Omega\}$$

which is almost surely linear, i.e. $\forall f, g \in T, \forall \lambda \in \mathbb{R}$

$$\varphi(f+g) = \varphi(f) + \varphi(g)$$
$$\varphi(\lambda f) = \lambda\varphi(f)$$

and which is continuous in the sense that if $f_n \to f$ in T then $\varphi(f_n) \to \varphi(f)$ in probability.

On the formal level, we have

$$\varphi(f) = \langle\varphi, f\rangle = \sum_{\alpha=1}^{N} \langle\varphi_\alpha, f_\alpha\rangle = \sum_{\alpha=1}^{N} \int_{\mathbb{R}^D} \varphi_\alpha(x) f_\alpha(x)\,\mathrm{d}x.$$

Definition 2.7. Let $\mathcal{D} = \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$ denote the space of test functions $f = (f_1, \ldots, f_N)$ with f_α being smooth test functions from $\mathbb{R}^D$ into $\mathbb{R}$ with compact supports. White noise is a generalized random field φ indexed by $\mathcal{D}$ such that its characteristic functional is given by

$$\Gamma(f) = E(\mathrm{e}^{\mathrm{i}\varphi(f)}) = \mathrm{e}^{-\int_{\mathbb{R}^D} \psi(f(x))\,\mathrm{d}x}. \tag{16}$$

The function $\psi : \mathbb{R}^N \to \mathbb{C}$ has the so-called Lévy–Khinchin representation

$$\psi(y) = \mathrm{i}\langle\beta, y\rangle + \frac{1}{2}\langle y, Ay\rangle + \int_{\mathbb{R}^N\setminus\{0\}} \left(1 - \mathrm{e}^{\mathrm{i}\langle\alpha, y\rangle} + \frac{\mathrm{i}\langle\alpha, y\rangle}{1+\|\alpha\|^2}\right) \frac{1+\|\alpha\|^2}{\|\alpha\|^2}\,\mathrm{d}\kappa(\alpha) \tag{17}$$

where $\beta \in \mathbb{R}^N$, A is a non-negative definite $N \times N$-matrix and κ is a non-negative, bounded measure on $\mathbb{R}^N \setminus \{0\}$ (see e.g. [6] for more details).

If $\kappa = 0$ and $A \neq 0$, φ is called Gaussian white noise whereas in the case $A = 0$, $\kappa \neq 0$, φ is called Poisson noise. In the following we always put $\beta = 0$ for simplicity.

In the last section we mentioned that we need representations of the group $\boldsymbol{G}$ in terms of real matrices. The reason for this is that $\mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$ is a vector space over $\mathbb{R}$.

Since $\mathcal{D}$ is a nuclear space, by Minlos' theorem [18] there is a unique probability measure μ on the dual space $\mathcal{D}'$ such that

$$\int_{\mathcal{D}'} \mathrm{e}^{\mathrm{i}(\eta, f)} \mathrm{d}\mu(\eta) = \Gamma(f)$$

where $(\cdot, \cdot)$ denotes the canonical pairing between $\mathcal{D}'$ and $\mathcal{D}$.

The function ψ in (17) is a negative definite function, [10].

$$\psi_{\mathrm{G}}(y) = \tfrac{1}{2}\langle y, Ay\rangle$$

is the Gaussian part and

$$\psi_{\mathrm{P}}(y) = \int_{\mathbb{R}^N \setminus \{0\}} \left(1 - \mathrm{e}^{\mathrm{i}\langle\alpha, y\rangle} + \frac{\mathrm{i}\langle\alpha, y\rangle}{1 + \|\alpha\|^2}\right) \frac{1 + \|\alpha\|^2}{\|\alpha\|^2} \,\mathrm{d}\kappa(\alpha)$$

is the Poisson part of ψ.

We shall also use the notation

$$\Gamma_{\mathrm{G}}(f) = E_{\mathrm{G}}(\mathrm{e}^{\mathrm{i}\varphi_{\mathrm{G}}(f)}) = \mathrm{e}^{-\int \psi_{\mathrm{G}}(f(x))\,\mathrm{d}x}$$

and the analogous notation for the Poisson part.

The noise φ can be regarded as the sum of Gaussian and Poisson noise: $\varphi = \varphi_{\mathrm{G}} + \varphi_{\mathrm{P}}$. Correspondingly, we have a measure μ_{G} and a measure μ_{P} on $\mathcal{D}'$, and μ is the convolution of these two measures: $\mu = \mu_{\mathrm{G}} * \mu_{\mathrm{P}}$.

Let us mention two characteristic properties of white noise. White noise is invariant under translations in the sense that the random variables $\varphi(f_{x_0})$ and $\varphi(f)$ are equal in law, where f_{x_0} is the function $x \mapsto f(x + x_0)$.

If we take two functions $f_1, f_2 \in \mathcal{D}$ with disjoint supports, the random variables $\varphi(f_1)$ and $\varphi(f_2)$ are independent.

If φ is white noise such that the random variables $\varphi(f)$ have zero mean and finite second moments $\forall f$, the function ψ in (17) has the so-called Kolmogorov canonical representation

$$\psi(y) = \tfrac{1}{2}\langle y, Ay\rangle + \int_{\mathbb{R}^N \setminus \{0\}} (1 - \mathrm{e}^{\mathrm{i}\langle\alpha, y\rangle} + \mathrm{i}\langle\alpha, y\rangle)\,\mathrm{d}\nu(\alpha) \tag{18}$$

where the so-called Lévy measure ν has the property $\int_{\mathbb{R}^N \setminus \{0\}} \|\alpha\|^2 \,\mathrm{d}\nu(\alpha) < \infty$. In this case ψ satisfies the inequality $|\psi(y)| \leqslant M\|y\|^2 \forall y \in \mathbb{R}^N$ where M is some constant $\geqslant 0$. This makes it possible to extend the generalized random field φ to L^2.

In the following we shall restrict the class of admitted characteristic functionals even further. We shall assume that the Lévy measure ν in (18) is invariant under the reflection $\alpha \mapsto -\alpha$. Under this assumption the characteristic functional corresponding to the Poisson part is of the form

$$\Gamma_{\mathrm{P}}(f) = E_{\mathrm{P}}(\mathrm{e}^{\mathrm{i}\varphi(f)}) = \mathrm{e}^{\int_{\mathbb{R}^D} \int_{\mathbb{R}^N} (\mathrm{e}^{\mathrm{i}\langle\alpha, f(x)\rangle} - 1)\,\mathrm{d}\nu(\alpha)\,\mathrm{d}x}. \tag{19}$$

Moreover, we assume that the measure ν satisfies the condition

$$\int_{\mathbb{R}^N} \mathrm{e}^{t\|\alpha\|} \,\mathrm{d}\nu(\alpha) < \infty \qquad \forall t \geqslant 0. \tag{20}$$

This condition guarantees the existence of all moments of the corresponding noise and, moreover, it allows us to extend the characteristic functional as an analytic function. To be more precise, for fixed $f \in \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$

$$\mathbb{C}^N \ni \xi \longmapsto \Gamma_{\mathrm{P}}(\xi f) = E_{\mathrm{P}}(\mathrm{e}^{\mathrm{i}\langle\cdot,\xi f\rangle}) \tag{21}$$

is an entire function in ξ obeying the estimate

$$|\Gamma_{\mathrm{P}}(\xi F)| \leqslant \exp(|\xi| \int \|f(x)\|\,\mathrm{d}x \int \|\alpha\| \mathrm{e}^{(|\xi|\|\alpha\|\|\|f\|\|)}\,\mathrm{d}\nu(\alpha)) \tag{22}$$

where $\|f(x)\| = (\sum_{i=1}^N |f_i(x)|^2)^{1/2}$ and $\|\|f\|\| = \sup_x |f(x)|$.

Lemma 2.9. Let us assume that the Lévy measure ν in (19) has finite first-order moments. Then for any $f \in D(\mathbb{R}^D)$, any cylinder function $F \in L^2(\mu_{\mathrm{P}})$ which is bounded and C^1 the following integration-by-parts formula holds

$$\int_{\mathcal{D}'(\mathbb{R}^D)\otimes\mathbb{R}^N} \langle \eta, f^\lambda \rangle F(\eta)\,\mathrm{d}\mu(\eta) = \int \left\langle f^\lambda(x), E\left(A\frac{\delta}{\delta\eta(x)}F(\eta)\right)\right\rangle \mathrm{d}x$$
$$+ \int\int f^\lambda(x) E(F(\eta + \alpha\delta(x - \cdot))\alpha_\lambda\,\mathrm{d}\nu(\alpha)\,\mathrm{d}x \tag{23}$$

where $(f^\lambda)_i = \delta_i^\lambda f$, $\frac{\delta}{\delta\eta(x)}$ denotes the functional derivative (widely used in mathematical physics, see e.g. [24]), and $(A\frac{\delta}{\delta\eta(x)})_j = \sum_{k=1}^N A_{jk}\frac{\delta}{\delta\eta_k(x)}$.

Proof. Take $F(\eta) = \exp \mathrm{i}(\eta, g)$. Employing (19), it is easily seen that (23) holds. Since any bounded C^1 cylinder function can be uniformly approximated by the sums $\sum_n c_n \exp \mathrm{i}(\eta, g)$ (see e.g. [24]) the assertion follows. □

If the characteristic functional of Poisson noise φ is of the form (19), the moments of φ are given by

$$E_{\mathrm{P}}\left(\prod_{i=1}^n (\varphi, f_i)\right) = \sum_{\substack{\Pi_1\cup\ldots\cup\Pi_k = J_n \\ \Pi_\alpha\cap\Pi_\beta=\emptyset \\ \text{for}\,\alpha\neq\beta}} \prod_{l=1}^k \int\int \prod_{j\in\Pi_l} \langle \alpha_l, f_j(x_l)\rangle\,\mathrm{d}x_l\,\mathrm{d}\nu(\alpha_l) \tag{24}$$

where the summation runs over the set of all partitions of $J_n = \{1, 2, \ldots, n\}$.

If the noise φ is the sum of a Gaussian and a Poisson part, formula (24) has to be altered:

$$E\left(\prod_{i=1}^n (\varphi, f_i)\right) = \sum_{\substack{\Pi_{\mathrm{G}}\cup\Pi_{\mathrm{P}} = J_n \\ \Pi_{\mathrm{G}}\cap\Pi_{\mathrm{P}}=\emptyset}} E_{\mathrm{P}}\left(\prod_{i\in\Pi_{\mathrm{P}}} (\varphi, f_i)\right) E_{\mathrm{G}}\left(\prod_{i\in\Pi_{\mathrm{G}}} (\varphi, f_i)\right). \tag{25}$$

The moments of the Gaussian part are uniquely determined by the covariance A:

$$E_{\mathrm{G}}((\varphi, f_1)(\varphi, f_2)) = \int \langle f_1(x), Af_2(x)\rangle\,\mathrm{d}x. \tag{26}$$

Remark 2.10. The number of terms in (24) is $\Pi_n = \sum_{p=1}^n S_n(p)$, where $S_n(p)$ are the so-called Stirling numbers of the second kind. They are given explicitly by

$$S_n(p) = \frac{1}{p!}\sum_{j=0}^p (-1)^j \binom{p}{j} (p - j)^n. \tag{27}$$

Therefore the total number of terms in (25) is $\sum_{k=0}^n \binom{n}{k}\Pi_k$.

To derive (19) we made the assumption that the Lévy measure ν is invariant under the reflection $\alpha \mapsto -\alpha$. This implies that the contributions coming from partitions (Π_α) in (24) containing some Π_α with an odd number of elements vanish.

The following remark shows that the carrier set of Poisson noise is extremely small: it consists of locally finite linear combinations of delta distributions.

Remark 2.11. Let

$C_{lf}(\mathbb{R}^D) = \{\Lambda \subset \mathbb{R}^D | \Lambda \cap K \text{ is finite for every compact set } K\}$

i.e. C_{lf} is the set of 'locally finite configurations'. C_{lf} can be given a topology such that C_{lf} is a complete metrizable space, [29].

If $\Lambda \in C_{\mathrm{lf}}(\mathbb{R}^D)$, Λ obviously contains either a finite number of points or countably many points. Let us fix an enumeration of these points, i.e. $\Lambda = (x_1, x_2, \ldots)$, $x_i \in \mathbb{R}^D$. Take $\Gamma = (\gamma_1, \gamma_2, \ldots) \in (\mathbb{R}^N)^{|\Lambda|}$. We define

$$\delta(\Lambda, \Gamma)(x) = \sum_{\substack{x_i \in \Lambda \\ \gamma_i \in \Gamma}} \gamma_i \delta(x - x_i) \tag{28}$$

where $\forall f = (f_1, \ldots, f_N) \in \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$

$$(\gamma_i \delta(\cdot - x_i), f) = \sum_{k=1}^{N} (\gamma_i)_k f_k(x_i). \tag{29}$$

Adapting the argument in [29], it can be proved that the set

$$\mathcal{C} = \{\delta(\Lambda, \Gamma) | \Lambda \in C_{\mathrm{lf}}(\mathbb{R}^D), \Gamma \in (\mathrm{supp}\nu)^{|\Lambda|}\}$$

is a carrier set for μ_{P}, i.e. $\mu_{\mathrm{P}}(\mathcal{C}) = 1$.

Remark 2.12. Let Λ be an open subset of $\mathbb{R}^D$. We define the σ-algebra $\Sigma(\Lambda)$ as the minimal μ_{P}-complete σ-algebra generated by the random variables $(\cdot, f)$ with $f \in \mathcal{D}(\mathbb{R}^D)$ supported in Λ. For Λ closed we define $\Sigma(\Lambda)$ as the intersection of all $\Sigma(\Lambda')$ where Λ' is open and $\Lambda \subset \Lambda'$. Let $\Gamma \subset \mathbb{R}^D$ be a closed subset of $\mathbb{R}^D$ of Lebesgue measure zero. It can be easily deduced from remark 2.11 that in this case $\Sigma(\Gamma)$ is trivial. It follows that the random field corresponding to μ_{P} is Markov in the following sense.

For any open $\Lambda \subset \mathbb{R}^D$ with sufficiently regular boundary $\partial\Lambda$ and any bounded F, G measurable with respect to $\Sigma(\Lambda)$ respectively $\Sigma(\Lambda^c)$

$$E_{\mu_{\mathrm{P}}}\{F \cdot G | \Sigma(\partial\Lambda)\} = E_{\mu_{\mathrm{P}}}\{F | \Sigma(\partial\Lambda)\} \cdot E_{\mu_{\mathrm{P}}}\{G | \Sigma(\partial\Lambda)\} = E_{\mu_{\mathrm{P}}}(F) \cdot E_{\mu_{\mathrm{P}}}(G) \tag{30}$$

where $E_{\mu_{\mathrm{P}}}\{-|\Sigma(\cdot)\}$ denotes the corresponding conditional expectation of $(-)$ with respect to the σ-algebra $\Sigma(\cdot)$.

Definition 2.8. Let τ be a representation of the group $(\boldsymbol{S})\boldsymbol{O}(\boldsymbol{D})$ in $\mathrm{Aut}\mathbb{R}^N$. We shall say that the random field φ given by (16) and (17) is τ-covariant iff

$$E(\mathrm{e}^{\mathrm{i}(\varphi, T_\tau f)}) = E(\mathrm{e}^{\mathrm{i}(\varphi, f)}) = E(\mathrm{e}^{\mathrm{i}(T_\tau^* \varphi, f)}) \tag{31}$$

for all $f \in \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$. T_τ^* is the adjoint of the T_τ with respect to the canonical pairing of $\mathcal{D}'$ and $\mathcal{D}$.

Lemma 2.13. Let τ be a representation of $(\boldsymbol{S})\boldsymbol{O}(\boldsymbol{D})$ in $\mathrm{Aut}\mathbb{R}^N$ and let φ be white noise given by (16) and (17). Then the noise φ is τ-covariant iff

(i) $\beta = 0$,
(ii) $\tau^{\mathrm{T}} A \tau = A$,
and
(iii) the measure $\mathrm{d}\kappa$ is τ-invariant, provided that τ is given by orthogonal matrices.

Let φ be τ-covariant white noise and let $R_\tau = \tau(R)$ be the representation of the reflection operator R in the representation τ, see (14). Let $f^\alpha \in \mathcal{D}(\mathbb{R}^D) \otimes \mathbb{R}^N$ be a finite sequence of test functions supported in $\{(t,x) \in \mathbb{R}^D | t > 0\}$. Then for any finite sequence $c_\alpha \in \mathbb{C}$ we have

$$\sum_{\alpha,\beta} c_\alpha \overline{c_\beta} \Gamma(\mathrm{e}^{\mathrm{i}\langle\varphi, f^\alpha\rangle} \mathrm{e}^{-\mathrm{i}\langle\varphi, R_\tau f^\beta\rangle}) = \sum_{\alpha,\beta} c_\alpha \overline{c_\beta} \Gamma(\mathrm{e}^{\mathrm{i}\langle\varphi, f^\alpha\rangle}) \Gamma(\mathrm{e}^{-\mathrm{i}\langle\varphi, R_\tau f^\beta\rangle}) = \left| \sum_\alpha c_\alpha \Gamma(\mathrm{e}^{\mathrm{i}\langle\varphi, f^\alpha\rangle}) \right|^2 \geqslant 0 \tag{32}$$

provided that the noise is R_τ-invariant.

Remark 2.14. The last property expresses the so-called reflection positivity of the noise φ. A covariant quantum field fulfilling all Wightman axioms can be constructed from the moments of such covariant reflection-positive noise, see e.g. [24, 43]. However, it is fairly easy to show that the arising quantum-field-theory operator is a multiple of the identity operator.

2.3. Covariant SPDEs and their solutions

Let $\mathcal{D} \in \mathrm{Cov}(\tau, \mathbb{R}^N)$ for some real representation τ of $\boldsymbol{SO(D)}$ and let $\tilde{\mathcal{D}}$ be the adjoint of $\mathcal{D}$ with respect to the canonical pairing of $\mathcal{S}'$ and $\mathcal{S}$ where $\mathcal{S}$ is the Schwartz space and $\mathcal{S}'$ is its topological dual. We shall consider SPDEs of the type

$$\tilde{\mathcal{D}}\varphi = \eta \tag{33}$$

where η is a given generalized random field indexed by $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$. An operator $\mathcal{D}$ will be called regular (correspondingly, the equation will be called regular) iff there exists a nuclear space $\mathcal{F}$ such that the Green function $\mathcal{D}^{-1}$ of $\mathcal{D}$ is defined on $\mathcal{F}$ and $\mathcal{D}^{-1}$ maps $\mathcal{F}$ continuously into $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$. A generalized random field φ indexed by $\mathcal{F}$ is called a weak solution of the regular equation (33) iff

$$\langle \varphi, f \rangle \cong \langle \eta, \mathcal{D}^{-1} f \rangle \qquad \text{for all } f \in \mathcal{F} \tag{34}$$

where $\cong$ means equality in law. Let Γ_η denote the characteristic functional of the field η. The characteristic functional Γ_φ of a weak solution φ of the regular equation (33) is given by

$$\Gamma_\varphi(f) = \Gamma_\eta(\mathcal{D}^{-1} f) \qquad \text{for } f \in \mathcal{F}. \tag{35}$$

If $\mathcal{D} : \mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N \to \mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$ is a continuous bijection $\mathcal{D}$ will be called strongly regular. For example, if $\mathcal{D}$ is admissible with strictly positive mass spectrum $\mathcal{D}$ is strongly regular. In the case of strongly regular $\mathcal{D}$ the space $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$ can be chosen as index space $\mathcal{F}(\mathbb{R}^D)$.

Even if the covariant operator $\mathcal{D}$ is invertible on the space $\mathcal{S}$, it may have a nontrivial kernel on $\mathcal{S}'$. Let $\mathcal{K}_\mathcal{D} \equiv \{\chi \in \mathcal{S}'(\mathbb{R}^D) \otimes \mathbb{R}^N | \ \tilde{\mathcal{D}}\chi = 0\}$. Then for any weak solution φ of a regular equation (33) and for any $\chi \in \mathcal{K}_\mathcal{D} \cap \mathcal{F}'$ the new random field φ_χ with characteristic functional

$$\Gamma_{\varphi_\chi}(f) = \int \mathrm{e}^{\mathrm{i}\langle\chi, f\rangle} \Gamma_\varphi(f) \, \mathrm{d}\nu(\chi) \tag{36}$$

is again a weak solution of (33). ν is a probability measure on $\mathcal{K}_\mathcal{D} \cap \mathcal{F}'$. In fact it can be proved that, fixing the space $\mathcal{F}$, every weak solutions of (33) is of the form (36).

Let us recall that a generalized random field η indexed by a space $\mathcal{F}$ is called τ-covariant iff

(i) T_g^τ acts in the space $\mathcal{F}$, and (ii) $\langle \eta, T_g^\tau f \rangle \cong \langle \eta, f \rangle$ for each $g \in (\boldsymbol{S})\boldsymbol{O}(\boldsymbol{D})$ and $f \in \mathcal{F}$.

Proposition 2.9. Let us consider equation (33) with regular $\mathcal{D}$ where η is τ-covariant. Then the weak solution of (33) given by (34) is again τ-covariant, provided that T_g^τ acts in the space $\mathcal{F}$.

Proof. Covariance implies that $\mathcal{D}T_g^\tau = T_g^\tau\mathcal{D}$. It follows easily that $\mathcal{D}^{-1}T_g^\tau = T_g^\tau\mathcal{D}^{-1}$. Therefore

$$\langle\varphi, T_g^\tau f\rangle \cong \langle\eta, \mathcal{D}^{-1}T_g^\tau f\rangle \cong \langle\eta, T_g^\tau\mathcal{D}^{-1} f\rangle \cong \langle\varphi, f\rangle.$$

□

In the massless case we can again consider equations of the type (33), where the covariant operator $\mathcal{D}$ now intertwines two representations, i.e. $\mathcal{D} \in \mathrm{Cov}((\mathbb{R}^N, \tau); (\mathbb{R}^N, \sigma))$. The notions of regularity and weak solution are defined analogously.

Proposition 2.10. Let $\mathcal{D} \in \mathrm{Cov}((\mathbb{R}^N, \tau), (\mathbb{R}^N, \sigma))$ be a regular operator and let η be a σ-covariant random field indexed by $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$. Then the weak solution of the equation

$$\tilde{\mathcal{D}}\varphi = \eta \tag{37}$$

given by (34) is a τ-covariant random field, provided that the corresponding index space $\mathcal{F}$ is T_g^τ-invariant.

Remark 2.15. It can be proved that in the case of irreducible representations of $\boldsymbol{SO}(\mathbf{4})$ the set $\mathrm{Cov}(\tau, \mathbb{K}^N)$ consists of zero-order operators only. Therefore, in order to construct nontrivial random fields, one has to consider operators which intertwine two representations, i.e. one puts $m = 0$ in equation (3) and takes some $\mathcal{D} \in \mathrm{Cov}((\mathbb{R}^N, \tau), (\mathbb{R}^N, \sigma))$. This has been done in the paper by Albeverio *et al* [4]. They studied the quaternionic Cauchy–Riemann operator $\partial \in \mathrm{Cov}((\mathbb{R}^N, \tau), (\mathbb{R}^N, \sigma))$ where $\tau = (\frac{1}{2}, \frac{1}{2})$ and $\sigma = (0, 1)$ are two reducible representations of $\boldsymbol{SO}(\mathbf{4})$.

For more details and new examples in $D = 4$ we refer to our forthcoming paper [19].

Remark 2.16. Let η be a τ-covariant generalized random field indexed by $\mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$ and let for simplicity $\mathcal{D}_1, \dots, \mathcal{D}_n, \dots \in \mathrm{Cov}(\tau, \mathbb{R}^N)$ be strongly regular. Let us consider the following sequence of covariant SPDEs:

$$\tilde{\mathcal{D}}_n\varphi^n = \varphi^{(n-1)} \qquad \tilde{\mathcal{D}}_1\varphi^1 = \eta \qquad \text{for } n = 1, 2, 3, \dots. \tag{38}$$

Then weak solutions φ^n of (37), provided they exist, give rise to a sequence (φ^n) of τ-covariant generalized random fields. In particular we have

$$\Gamma_{\varphi^{(n)}}(f) = \Gamma_\eta(\mathcal{D}_n^{-1} \dots \mathcal{D}_1^{-1} f). \tag{39}$$

Let $\mathcal{S}(\mathbb{R}_{+/(-)} \otimes \mathbb{R}^{D-1}) = \{f \in \mathcal{S}(\mathbb{R}^D) | \mathrm{supp} f \subset \{x_0 > (<) 0, \boldsymbol{x} \in \mathbb{R}^{D-1}\}$.

Let $\mathcal{R} : \mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N \to \mathcal{S}(\mathbb{R}^D) \otimes \mathbb{R}^N$ be a continuous linear map such that

(i) $\mathcal{R} : \mathcal{S}(\mathbb{R}^D_{+/(-)}) \otimes \mathbb{R}^N \to \mathcal{S}(\mathbb{R}^D_{-/(+)}) \otimes \mathbb{R}^N$

(ii) $\mathcal{R}^2 = \mathrm{id}$.

A given random field η is called $\mathcal{R}$-reflection positive iff for all finite sequences $c_k \in \mathbb{C}$, $f_k \in \mathcal{S}(\mathbb{R}^D_+) \otimes \mathbb{R}^N$ the following inequality holds:

$$\sum_{k,l} c_k \bar{c}_l \Gamma_\eta(f_k - \mathcal{R} f_l) \geqslant 0. \tag{40}$$

Proposition 2.11. Let $\mathcal{D} \in \mathrm{Cov}(\tau, \mathbb{R}^N)$ be strongly regular and let η be $\mathcal{R}$-reflection positive. Define $\mathcal{F}_{+/(-)} \equiv \{f \in \mathcal{F} | \exists g \in \mathcal{S}(\mathbb{R}^D_{+/(-)}) \otimes \mathbb{R}^N$ such that $f = \mathcal{D}g$ and $[\mathcal{R}, \mathcal{D}]g = 0\}$. Then the weak solution φ of (33) given by (34) is $\mathcal{R}$-reflection positive in the following sense:

For all finite sequences $c_k \in \mathbb{C}$, $f_k \in \mathcal{F}_+$ we have

$$\sum_{k,l} c_k \bar{c}_l \Gamma_\varphi(f_k - \mathcal{R} f_l) \geqslant 0.$$

Proof. Let $f_k = \mathcal{D} g_k$, where $g_k \in \mathcal{S}(\mathbb{R}_+ \otimes \mathbb{R}^{D-1}) \otimes \mathbb{R}^N$. We use $\mathcal{R}$-reflection positivity (40) of η:

$$\sum_{k,l} c_k \bar{c}_l \Gamma_\varphi(f_k - \mathcal{R} f_l) = \sum_{k,l} c_k \bar{c}_l \Gamma_\eta(\mathcal{D}^{-1} f_k - \mathcal{D}^{-1} \mathcal{R} f_l) = \sum_{k,l} c_k \bar{c}_l \Gamma_\eta(g_k - \mathcal{R} g_l) \geqslant 0.$$

□

Remark 2.17. We emphasize that in the previous proposition we have reflection positivity only on the subspace $\mathcal{D}(\mathcal{S}(\mathbb{R}^D_+ \otimes \mathbb{R}^N))$. Though this subspace might look temptingly big, it is too small to produce nontrivial models. Consider the equation $\tilde{\mathcal{D}}\varphi = \eta$, where η is white noise, and take $f \in \mathcal{D}(\mathcal{S}(\mathbb{R}^D_+ \otimes \mathbb{R}^N))$. The scalar product in the physical Hilbert space is the trivial one given by white noise:

$$\langle \varphi(\mathcal{R} f), \varphi(f) \rangle = \langle \eta(\mathcal{D}^{-1} \mathcal{R} f), \eta(\mathcal{D}^{-1} f) \rangle = \langle \eta(\mathcal{R} g), \eta(g) \rangle.$$

A detailed discussion of reflection positivity for higher-spin bosonic models of Euclidean quantum-field theory together with a proof of the no-go theorem quoted in the introduction can be found in [8, 9, 20].

From now on we specialize our discussion to the case when η is τ-covariant white noise with characteristic functional Γ_η given by $\Gamma_\eta = \Gamma^{\mathrm{G}}_\eta \Gamma^{\mathrm{P}}_\eta$ where Γ^{G}_η is given by the Gaussian part of (18) and Γ^{P}_η is given by (19). We collect some elementary properties of the weak solution of (33).

(1) The weak solution φ of a regular equation (33) with $\mathcal{D} \in \mathrm{Cov}(\tau, \mathbb{R}^N)$ has the characteristic functional Γ_φ

$$\mathcal{F} \ni f \to \Gamma^{\mathrm{C}}_\varphi = \Gamma^{\mathrm{G}}_\varphi \Gamma^{\mathrm{P}}_\varphi(f) \tag{41}$$

where

$$\Gamma^{\mathrm{G}}_\varphi = \mathrm{e}^{-\frac{1}{2}\int \langle f(x), (\mathcal{D}^{-1})^{\mathrm{T}} A \mathcal{D}^{-1}(x-y) f(x) \rangle \, \mathrm{d}x \, \mathrm{d}y} \tag{42}$$

$$\Gamma^{\mathrm{P}}_\varphi = \mathrm{e}^{\int \int [\mathrm{e}^{\mathrm{i}\langle \alpha, \mathcal{D}^{-1} * f \rangle (x)} - 1] \, \mathrm{d}\nu(\alpha) \, \mathrm{d}x}. \tag{43}$$

There exists a unique Borel cylindric probability measure $\mathrm{d}\mu_{\mathcal{D}}(\varphi)$ on $\mathcal{F}'(\mathbb{R}^D)$($\equiv$ the weak dual of $\mathcal{F}$) such that

$$\Gamma_\varphi(f) \equiv \int_{\mathcal{F}'(\mathbb{R}^D)} \mathrm{d}\mu_{\mathcal{D}}(\varphi) \mathrm{e}^{\mathrm{i}\langle \varphi, f \rangle}. \tag{44}$$

(2) For any bounded C^1 cylindric function $F \in L^2(\mathrm{d}\mu_{\mathcal{D}})$ the following integration-by-parts formula holds:

$$\int_{\mathcal{F}'(\mathbb{R}^D) \otimes \mathbb{R}^N} \langle \varphi, f^\lambda \rangle F(\varphi) \, \mathrm{d}\mu_{\mathcal{D}}(\varphi) = \int \left\langle f^\lambda(x), E\left((\mathcal{D}^{-1})^{\mathrm{T}} A \mathcal{D}^{-1} \frac{\delta}{\delta \varphi(x)} F(\varphi) \right) \right\rangle \mathrm{d}x$$
$$+ \int \int f(x) E F(\varphi + \alpha_\lambda \tilde{\mathcal{D}}^{-1}(\cdot - x)) \alpha_\lambda \, \mathrm{d}\nu(\alpha) \, \mathrm{d}x \tag{45}$$

where $(f^\lambda)_i = \delta_i^\lambda f$, $f \in \mathcal{F}$, δ_i^τ is Kronecker delta.

(3) If the Levy measure dν has all moments then the field φ has all moments and they are given by the following formula:

$$E\left(\prod_{i=1}^{n}(\varphi, f_i)\right) = \sum_{\substack{\Pi_G \cup \Pi_P = J_n \\ \Pi_G \cap \Pi_P = \emptyset}} E_P\left(\prod_{i\in\Pi_P}(\varphi, f_i)\right) E_G\left(\prod_{i\in\Pi_G}(\varphi, f_i)\right) \tag{46}$$

where

$$E_P\left(\prod_{i=1}^{n}(\varphi, f_i)\right) = \sum_{\substack{\Pi_1\cup\ldots\cup\Pi_k = J_n \\ \Pi_\alpha\cap\Pi_\beta=\emptyset \\ \text{for } \alpha\neq\beta}} \prod_{l=1}^{k} \int \ldots \int \prod_{j\in\Pi_l} \langle \alpha_l, \mathcal{D}^{-1} f_j(x_l)\rangle \, \mathrm{d}x_l \, \mathrm{d}\nu(\alpha_l) \tag{47}$$

and

$$E_G\left(\prod_{i=1}^{2n}(\varphi, f_i)\right) = \sum_{\substack{i_k<j_k \\ k=1,\ldots,n}} \prod_{l=1}^{k} \iint \mathrm{d}x\, \mathrm{d}y\, \langle f_{i_k}(x), (\mathcal{D}^{-1})^{\mathrm{T}} A \mathcal{D}^{-1}(x-y) f_{j_k}(x)\rangle \tag{48}$$

$$E_G\left(\prod_{i=1}^{2n+1}(\varphi, f_i)\right) = 0. \tag{49}$$

In particular the two-point moment $S_\varphi^2 \in \mathcal{F}'^{\otimes 2}$ of φ is given by

$$S_\varphi^2(f\otimes g) = (\mathcal{D}^{-1})^{\mathrm{T}} A \mathcal{D}^{-1}(f\otimes g) + \int \mathrm{d}\nu\,(\alpha) \int \mathrm{d}x\, \langle \alpha, \mathcal{D}^{-1} f(x)\rangle \langle \alpha, \mathcal{D}^{-1} g(x)\rangle \tag{50}$$

which has the following kernel

$$S_\varphi^2(x-y) = (\mathcal{D}^{-1})^{\mathrm{T}} A \mathcal{D}^{-1}(x-y) + \int \mathrm{d}\nu\,(\alpha) \int \mathrm{d}z\, \langle \alpha, \mathcal{D}^{-1}(z-x)\rangle \langle \alpha, \mathcal{D}^{-1}(z-y)\rangle. \tag{51}$$

(4) The set $\tilde{\mathcal{D}}^{-1} * C \equiv \{\sum_i \alpha_i \tilde{\mathcal{D}}^{-1}(\cdot - x) |$ where $\{x_i\} \in C_{if}(\mathbb{R}^D)$ and $\alpha_i \in \operatorname{supp} \mathrm{d}\nu$ for all $i\}$ is the carrier set of the Poisson part of the measure $\mathrm{d}\mu_{\mathcal{D}}$, see remark 2.11.

(5) If the noise is τ-covariant then the random field φ is τ-covariant, provided that the test function space $\mathcal{F}$ is T^τ-invariant.

(6) In the case of a strongly regular equation the corresponding solution is Markov. The preservation of the Markov property under the transformation $\eta \longrightarrow \mathcal{D}^{-1}\eta$ with $\det \hat{\mathcal{D}}(\mathrm{i}p) \neq 0$, $p \in \mathbb{R}^D$ follows immediately from [30]. The case of nontrivial ker$\mathcal{D}$ is more subtle [27, 34, 47]. The solutions of (33) with η being Gaussian lead to Gaussian solutions and are therefore not very interesting from the point of view of physics. This is why we require that the Poisson part of the white noise η is nonzero in all further applications.

Remark 2.18. Other fundamental properties of the field φ like Markov property, lattice approximation(s) will be discussed elsewhere (see e.g. [5, 27, 30, 36]).

3. Fourier–Laplace transform properties of the solutions

Let us define the following spaces of functions:

$\mathcal{S}_+(\mathbb{R}^{Dn}) = \{f \in \mathcal{S}(\mathbb{R}^{Dn}) | f$ and all its derivatives vanish unless $0 < x_1^0 < x_2^0 < \cdots < x_n^0\}$

$\mathcal{S}_0(\mathbb{R}^{Dn}) = \{f \in \mathcal{S}(\mathbb{R}^{Dn}) | f$ and all its derivatives vanish if $x_i = x_j$ for some $1 \leqslant i < j \leqslant n\}$

$\mathcal{S}(\mathbb{R}_+) = \{f \in \mathcal{S}(\mathbb{R}) | \operatorname{supp} f \subseteq [0, \infty)\}$ $\qquad$ $\mathcal{S}(\mathbb{R}_-) = \{f \in \mathcal{S}(\mathbb{R}) | \operatorname{supp} f \subseteq (-\infty, 0]\}$.

We identify the following spaces:

$$\mathcal{S}(\overline{\mathbb{R}_+}) = \mathcal{S}(\mathbb{R})/\mathcal{S}(\mathbb{R}_-) \qquad \mathcal{S}(\overline{\mathbb{R}_+}^D) = \mathcal{S}(\overline{\mathbb{R}_+}) \otimes \mathcal{S}(\mathbb{R}^{D-1})$$
$$\mathcal{S}(\mathbb{R}^D;\mathbb{R}^N) = \mathbb{R}^N \otimes \mathcal{S}(\mathbb{R}^D)$$
$$\mathcal{S}(\mathbb{R}^{Dn};(\mathbb{R}^N)^{\otimes n}) = (\mathbb{R}^N)^{\otimes n} \otimes \mathcal{S}(\mathbb{R}^{Dn})$$
$$\mathcal{S}_+(\mathbb{R}^{Dn};(\mathbb{R}^N)^{\otimes n}) = (\mathbb{R}^N)^{\otimes n} \otimes \mathcal{S}_+(\mathbb{R}^{Dn})$$
$$\mathcal{S}_0(\mathbb{R}^{Dn};(\mathbb{R}^N)^{\otimes n}) = (\mathbb{R}^N)^{\otimes n} \otimes \mathcal{S}_0(\mathbb{R}^{Dn})$$
$$\mathcal{S}((\overline{\mathbb{R}_+}^D)^n;(\mathbb{R}^N)^{\otimes n}) = (\mathbb{R}^N)^{\otimes n} \otimes \mathcal{S}(\overline{\mathbb{R}_+}^{Dn}).$$

The following maps will be used:

$$d : \mathcal{S}(\mathbb{R}^{Dn}) \ni f \mapsto f^d(x_1, x_2 - x_1, \ldots, x_n - x_{n-1}) \equiv f(x_1, \ldots, x_n). \tag{52}$$

The map d is a morphism of $\mathcal{S}_+(\mathbb{R}^{Dn};(\mathbb{R}^N)^{\otimes n})$ into $\mathcal{S}((\overline{\mathbb{R}_+}^D)^n;(\mathbb{R}^N)^{\otimes n})$. The Fourier–Laplace transform on $\mathcal{S}(\overline{\mathbb{R}_+}^{Dn})$ is

$$\mathcal{S}(\overline{\mathbb{R}_+}^{Dn}) \ni f_n \mapsto f_n^{\mathrm{FL}}(q_1, \ldots, q_n) = \int \mathrm{e}^{-\sum_{k=1}^n q_k^0 x_k^0} \mathrm{e}^{\mathrm{i}\sum_{k=1}^n \boldsymbol{q}_k \cdot \boldsymbol{x}_k} f_n(x_1, \ldots, x_n) \otimes_{l=1}^n \mathrm{d}x_l. \tag{53}$$

Finally, we have the map

$$\eta : \mathcal{S}_+(\mathbb{R}^{D(n+1)}) \ni f_n \mapsto \eta f_n \in \mathcal{S}(\overline{\mathbb{R}_+}^{Dn}) \tag{54}$$

where

$$\eta(f_n)(p_1, \ldots, p_n) \equiv f_n^{d,\mathrm{FL}}(p_1, \ldots, p_n)|_{\{p_k^0 \geqslant 0\}}. \tag{55}$$

It is well known [41] that the map η is continuous with dense range in $\mathcal{S}(\overline{\mathbb{R}}_+^{Dn})$ and trivial kernel. The notions of d, of taking the Fourier–Laplace transform and of the map η naturally extend to the case of distributions with multi-indices.

Definition 3.1. A distribution $F_{n+1} \in \mathcal{S}'_+(\mathbb{R}^{D(n+1)}, (\mathbb{R}^N)^{\otimes(n+1)})$ has the Fourier–Laplace property (the FL property) iff there exists a distribution $\mathcal{W}_n \in \mathcal{S}'(\overline{\mathbb{R}}_+^D, (\mathbb{R}^N)^{\otimes n})$ such that

$$F_{n+1}^d(x_0, \ldots, x_n) \equiv \int \mathrm{e}^{-\sum_{k=1}^n p_k^0 x_k^0} \mathrm{e}^{\mathrm{i}\sum_{k=1}^n \boldsymbol{p}_k \cdot \boldsymbol{x}_k} \mathcal{W}_n(p_1, \ldots, p_n)\, \mathrm{d}p_1 \ldots \mathrm{d}p_n \tag{56}$$

where the equality is understood in the sense of distributions, see e.g. [12, 40, 41].

There are several necessary and sufficient conditions which guarantee that a given $F_n \in \mathcal{S}'_+(\mathbb{R}^{Dn})$ has the FL property [12, 41, 43]. However, all known criteria are difficult to check in concrete situations.

Let τ be a representation of $\boldsymbol{SO(D)}$ in the space $\mathrm{Aut}(\mathbb{R}^N)$. We shall say that a tempered distribution $S_n \in \mathcal{S}'(\mathbb{R}^D;(\mathbb{R}^N)^{\otimes n})$ is covariant under the action of τ (τ-covariant) iff for each $g \in \boldsymbol{SO(D)}$ $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^D;\mathbb{R}^N)$ the following equality holds:

$$S_n(f_1 \otimes \cdots \otimes f_n) = S_n(T_{\tau_g} f_1 \otimes \cdots \otimes T_{\tau_g} f_n) \tag{57}$$

where $(T_{\tau_g} f)(x) \equiv \tau_g f(g^{-1}x)$. A distribution $S_n \in \mathcal{S}'(\mathbb{R}^D;(\mathbb{R}^N)^{\otimes n})$ is called symmetric iff

$$S_n(f_1 \otimes \cdots \otimes f_n) = S_n(f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}) \tag{58}$$

for any $\pi \in S^n$ ($\equiv$ permutation group) and any $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^D;\mathbb{R}^N)$.

Proposition 3.2. Let τ be a representation of the group $\boldsymbol{SO(D)}$ in $\mathrm{Aut}(\mathbb{R}^N)$. If $\sigma_n \in \mathcal{S}'(\mathbb{R}^D; (\mathbb{R}^N)^{\otimes n})$ is symmetric covariant under the action of τ and $\sigma_n|_{\{y_k^0>0\}}$ has FL property then there exists a unique tempered distribution $\mathcal{W}_n \in \mathcal{S}'(\mathbb{R}^D; (\mathbb{R}^N)^{\otimes n})$ such that:

(1) $\mathcal{W}_n^{\mathrm{F}}$ is supported in the product of forward light cones $V^+ \equiv \{p \in M^D | p \cdot p \geqslant 0;\ p^0 \geqslant 0\}$, i.e.

$$\mathrm{supp}\mathcal{W}_n \subseteq (V^+)^{\times n}$$

(2) $\mathcal{W}_n$ is covariant under the representation τ^M of $\boldsymbol{SO}(D-1,1)$, i.e.

$$\mathcal{W}_n(f_1 \otimes \cdots \otimes f_n) = \mathcal{W}_n(T_{\tau_g^M} f_1 \otimes \cdots \otimes T_{\tau_g^M} f_n) \tag{59}$$

for any $g \in L_+^\uparrow(D)$; $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^D; \mathbb{R}^n)$ and where τ^M is the analytic continuation of τ into therepresentation of $\boldsymbol{SO}(D-1,1)$ via the 'Weyl unitary trick',

(3) $\mathcal{W}_n$ is local which means that the inverse Fourier transform of $\mathcal{W}_n(x_1, \ldots, x_n)$ has the property that if some x_i, x_{i+1} are such that $(x_i - x_{i+1})^2 < 0$ then

$$\mathcal{W}_n(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) = \mathcal{W}_n(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n) \tag{60}$$

(4)

$$S_{n+1}^d(x_0, \ldots, x_n) \equiv \int \mathrm{e}^{-\sum_{k=1}^n p_k^0 x_k^0} \mathrm{e}^{\mathrm{i}\sum_{k=1}^n \boldsymbol{p}_k \cdot \boldsymbol{x}_k} \mathcal{W}_n^F(p_1, \ldots, p_n) \prod_{i=1}^n \mathrm{d}p_i \tag{61}$$

for $x_1^0 < \cdots < x_n^0$.

Proof. From the FL property of S_n it follows that there exists $\mathcal{W}_n \in \mathcal{S}'(\mathbb{R}^D; (\mathbb{R}^N)^{\otimes n})$ such that $\mathcal{W}_n$ is supported on positive energies, i.e. on the set $\{(p_1, \ldots, p_n) | p_i^0 \geqslant 0$ for all $i = 1, \ldots, n\}$ and such that (4) holds. However, the distribution which is covariant under the action of the Lorentz group must be supported in the orbit of Lorentz group [12, 28, 44] and thus we conclude that $\mathcal{W}_n$ must be supported in $(V^+)^{\times n}$. The locality of $\mathcal{W}_n$ follows from the symmetry property of S_n (see e.g. [28]). The uniqueness of $\mathcal{W}_n$ follows from the fact that the kernel of the Fourier–Laplace transform consists only of the vector 0. □

The difference variables moments σ_n of the random fields A constructed in section 2 are defined as

$$\sigma_n^A(\xi_1, \ldots, \xi_n) \equiv S_{n+1}^A(x_1, \ldots, x_{n+1}) \tag{62}$$

where $\xi_i \equiv x_{i+1} - x_i$ for $i = 1, \ldots, n$. Now we are ready to formulate the main result of this paper.

Theorem 3.1. Let τ be a real representation of $\boldsymbol{SO(D)}$ in $\mathrm{Aut}(\mathbb{R}^N)$, $\mathcal{D} \in \mathrm{Cov}(\tau, \mathbb{R}^N)$ with an admissible spectrum and let A be a solution of

$$\tilde{\mathcal{D}}A = \eta$$

where η is a T_τ-invariant Poisson noise. Then the difference variables moments $\sigma_n^A(x_1, \ldots, x_n)$ restricted to $0 < x_1^0, \ldots, 0 < x_n^0$ have the FL property.

The proof of this theorem will be divided into three main steps.

Proposition 3.3. Let $\mathcal{D} \in \mathrm{Cov}(\tau, \mathbb{R}^N)$ have an admissible mass spectrum with strictly positive masses. Then the Green function $G_A = (\mathcal{D}_A)^{-1*}$ of $\mathcal{D}$ restricted to $x_1^0 < x_2^0$ has the FL property.

Lemma 3.2. Let A, G_A be as in proposition 3.3. Then

$$S_2 = (|x_1 - x_2|) = \int \mathrm{d}x\, \mathcal{D}_A^{-1}(x - x_1)\mathcal{D}_A^{-1}(x - x_2) \tag{63}$$

restricted to $x_1^0 < x_2^0$ has the FL property.

Lemma 3.3. Let A, G_A be as in proposition 3.4. Then for any $k = 1, 2, \ldots$ the distribution S'_k, where

$$S'_k(x_1, \ldots, x_k) \equiv \int \mathrm{d}x\, \mathcal{D}_A^{-1}(x - x_1) \ldots \mathcal{D}_A^{-1}(x - x_k) \tag{64}$$

restricted to $x_1^0 < \cdots < x_k^0$ has the FL property.

The separation of the proof into lemma 3.2 and lemma 3.3 is made for reader's convenience only. Having proved proposition 3.1 and lemma 3.3, the proof of theorem 3.3 follows by observing that the FL property is stable under taking tensor products and using formula (47). The case that some of the masses are zero is easily covered by using the continuity of the Fourier–Laplace transform and an easily controlled limiting procedure: we artificially introduce small nonzero masses in the corresponding formulae, then let them tend to zero. Although the covariance might be broken by introducing these masses it can be restored in the limit.

Proof of proposition 3.3. The typical matrix element $G_A^{\alpha\beta}$ of G_A has the form

$$G_A^{\alpha\beta}(p) = \frac{Q^{\alpha\beta}(p)}{\prod_{i=1}^n (p_0^2 + \boldsymbol{p}^2 + m_i^2)} \tag{65}$$

where $Q^{\alpha\beta}$ are polynomials in the variables p of degree lower or equal $N-2$ and all $m_i > 0$ due to assumption made on $\mathcal{D}$.

Let us first assume that all the m_i in equation (65) are different. Regarding the right-hand side of (65) as a function of p_0 we can decompose it into elementary quotients

$$\frac{Q^{\alpha\beta}(p)}{\prod_{i=1}^n (p_0^2 + \boldsymbol{p}^2 + m_i^2)} = \sum_{i=1}^n \frac{Q_i^{\alpha\beta}(p_0, \boldsymbol{p})}{(p_0^2 + \boldsymbol{p}^2 + m_i^2)} \tag{66}$$

where $Q_i^{\alpha\beta}(p_0, \boldsymbol{p}) \equiv A^{\alpha\beta,i}(\boldsymbol{p})p_0 + B^{\alpha\beta,i}(\boldsymbol{p})$. $A^{\alpha\beta,i}(\boldsymbol{p})$, $B^{\alpha\beta,i}(\boldsymbol{p})$ are rational functions in the variable $\boldsymbol{p}$ which have no singularities on the real line. It is well known that the distribution

$$S^2_{\Box,i}(x) = \int \frac{1}{p_0^2 + \boldsymbol{p}^2 + m_i^2} \mathrm{e}^{-\mathrm{i}px}\, \mathrm{d}p \tag{67}$$

has the FL property with the underlying distribution $\mathcal{W}_i^0$ given by $\mathcal{W}_i^0(p_0, \boldsymbol{p}) = \epsilon(p_0)\delta(p_0^2 - \boldsymbol{p}^2 - m^2)$, where $\epsilon(p_0) = 1$ if $p_0 \geqslant 0$ and 0 otherwise. The inverse Fourier transform of a typical term appearing in (66) is given for $x^0 \geqslant 0$

$$\begin{aligned}
&\left(A_i^{\alpha\beta}(\mathrm{i}\nabla)\mathrm{i}\frac{\partial}{\partial x^0} + B_i^{\alpha\beta}(\mathrm{i}\nabla)\right) S^2_{\Box,i}(x^0, \boldsymbol{x}) \\
&\quad = (A_i^{\alpha\beta}(\mathrm{i}\nabla)\mathrm{i}\frac{\partial}{\partial x^0} + B_i^{\alpha\beta}(\mathrm{i}\nabla))\left(\int_0^\infty \int \mathrm{e}^{-p_0 x_0}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\delta(p_0^2 - \boldsymbol{p}^2 - m_i^2)\, \mathrm{d}p_0\, \mathrm{d}\boldsymbol{p}\right) \\
&\quad = \int_0^\infty \int \mathrm{e}^{-p_0 x_0}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\delta(p_0^2 - \boldsymbol{p}^2 - m_i^2)B_i^{\alpha\beta}(\boldsymbol{p})\, \mathrm{d}p_0\, \mathrm{d}\boldsymbol{p} \\
&\qquad + \int_0^\infty \int \mathrm{e}^{-p_0 x_0}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\{-\mathrm{i}p_0 A_i^{\alpha\beta}(\boldsymbol{p})\delta(p_0^2 - \boldsymbol{p}^2 - m_i^2)\}\, \mathrm{d}p_0\, \mathrm{d}\boldsymbol{p}
\end{aligned} \tag{68}$$

and this shows that the inverse Fourier transform of each term in (66) is the Fourier–Laplace transform with underlying distribution for the ith term

$$\mathcal{W}_i^{\alpha\beta}(p_0,\boldsymbol{p}) = \{B_i^{\alpha\beta}(\boldsymbol{p}) - \mathrm{i}p_0 A_i^{\alpha\beta}(\boldsymbol{p})\}\delta(p_0^2 - \boldsymbol{p}^2 - m_i^2)\epsilon(p_0) \tag{69}$$

and $\mathcal{W}_G^{\alpha\beta}(p_0,\boldsymbol{p}) \equiv \sum_i \mathcal{W}_i^{\alpha\beta}(p_0,\boldsymbol{p})$.

Let us now consider the case that some of the m_i in equation (65) are equal. In this case the right-hand side of (65) is a sum of terms of the following form

$$\frac{Q(p_0,\boldsymbol{p})}{(p_0^2+\boldsymbol{p}^2+m^2)^\alpha}$$

where the integer α is $\geqslant 1$. The FL property now follows from the above arguments and the following formula, which can be obtained by contour integration.

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}p_0x_0}}{(p_0^2+\omega^2)^\alpha}\,\mathrm{d}p_0 = \sum_{k=0}^{\alpha-1} A_k^\alpha |x_0|^{\alpha-1-k}\frac{\mathrm{e}^{-\omega|x_0|}}{\omega^{\alpha+k}}$$

where $\omega^2 = \boldsymbol{p}^2 + m^2$ and $A_k^\alpha \in \mathbb{R}$. This completes the proof of proposition 3.4. □

In the proofs of lemmas 3.2 and 3.3 we shall assume for notational simplification that all the m_i in equation (65) are different so that we can use expansion (66).

Proof of lemma 3.2. We shall proceed very close to the proof of theorem 4.21 in [4]. First, we use the following identity

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-\zeta_1|t-t_1|}\mathrm{e}^{-\zeta_2|t-t_2|}\,\mathrm{d}t = \frac{1}{\zeta_1+\zeta_2}\mathrm{e}^{-\zeta_2(t_2-t_1)} + \frac{1}{\zeta_1+\zeta_2}\mathrm{e}^{-\zeta_1(t_2-t_1)} + (t_2-t_1)\int_0^1 \mathrm{e}^{-(\zeta_1 s+(1-s)\zeta_2)(t_2-t_1)}\,\mathrm{d}s \tag{70}$$

which is valid for any $t_1, t_2 \in \mathbb{R}$, $\zeta_1, \zeta_2 \in \mathbb{C}$ such that $t_2 - t_1 > 0$, $\Re\zeta_1 > 0$, and $\Re\zeta_2 > 0$. Second we note that:

$$\int_{\mathbb{R}^{D-1}}\int_0^\infty \mathrm{e}^{-p_0|x_0|}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\mathcal{W}_i^{\alpha\beta}(p_0,\boldsymbol{p})\,\mathrm{d}p_0\,\mathrm{d}\boldsymbol{p} = \frac{-\mathrm{i}}{2}\int_{\mathbb{R}^{D-1}} \mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}|x_0|}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}A_i^{\alpha\beta}(\boldsymbol{p})\mathrm{d}^{D-1}\boldsymbol{p} + \frac{1}{2}\int_{\mathbb{R}^{D-1}} \frac{\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}|x_0|}}{\sqrt{\boldsymbol{p}^2+m_i^2}}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}B_i^{\alpha\beta}(\boldsymbol{p})\,\mathrm{d}^{D-1}\boldsymbol{p}. \tag{71}$$

Now, we have

$$\begin{aligned}
S(y_1-y_2) &\equiv \int \mathrm{d}x^0\,\mathrm{d}\boldsymbol{x}\,\mathcal{D}^{-1}(x-y_1)\mathcal{D}^{-1}(x-y_2)\\
&= \int \mathrm{d}x^0\,\mathrm{d}\boldsymbol{x}\,\mathcal{D}^{-1}(|x^0-y_1^0|,(\boldsymbol{x}-\boldsymbol{y}_1))\mathcal{D}^{-1}(|x^0-y_2^0|,(\boldsymbol{x}-\boldsymbol{y}_2))\\
&= \int_{-\infty}^{+\infty}\mathrm{d}x^0\int\mathrm{d}\boldsymbol{x}\int\mathrm{d}p_1\int\mathrm{d}p_2\,\mathrm{e}^{-p_1^0|x^0-y_1^0|}\mathrm{e}^{-p_2^0|x^0-y_2^0|}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}_1\cdot(\boldsymbol{x}-\boldsymbol{y}_1)}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}_2\cdot(\boldsymbol{x}-\boldsymbol{y}_2)}\\
&\quad\times\mathcal{W}_G(p_1^0,\boldsymbol{p}_1)\mathcal{W}_G(p_2^0,\boldsymbol{p}_2)\\
&= \int\mathrm{d}\boldsymbol{x}\int\mathrm{d}p_1\int\mathrm{d}p_2\left\{\frac{1}{p_1^0+p_2^0}(\mathrm{e}^{-p_1^0(y_2^0-y_1^0)}+\mathrm{e}^{-p_2^0(y_2^0-y_1^0)})\right.\\
&\quad\left.+(y_2^0-y_1^0)\int_0^1\mathrm{d}s\,\mathrm{e}^{-(p_1^0 s+(1-s)p_2^0)(y_2^0-y_1^0)}\right\}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}_1\cdot(\boldsymbol{x}-\boldsymbol{y}_1)}\mathrm{e}^{-\mathrm{i}\boldsymbol{p}_2\cdot(\boldsymbol{x}-\boldsymbol{y}_2)}\\
&\quad\times\mathcal{W}_G(p_1^0,\boldsymbol{p}_1)\mathcal{W}_G(p_2^0,\boldsymbol{p}_2).
\end{aligned} \tag{72}$$

Defining the following functions

$$\Pi_{(1)}^{\alpha\beta i,\alpha'\beta' i^*} = -\frac{1}{4}\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}_2-\boldsymbol{y}_1)}\frac{A_i^{\alpha\beta}(-\boldsymbol{p})A_{i^*}^{\alpha'\beta'}(\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+m_i^2}+\sqrt{\boldsymbol{p}^2+m_{i^*}^2}}\{\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}(y_2^0-y_1^0)} + \mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_{i^*}^2}(y_2^0-y_1^0)}\} \tag{73}$$

$$\Pi_{(2)}^{\alpha\beta i,\alpha'\beta' i^*} = \frac{1}{4}\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}_2-\boldsymbol{y}_1)}\frac{B_i^{\alpha\beta}(-\boldsymbol{p})B_{i^*}^{\alpha'\beta'}(\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+m_i^2}+\sqrt{\boldsymbol{p}^2+m_{i^*}^2}} \times\{\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}(y_2^0-y_1^0)} + \mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_{i^*}^2}(y_2^0-y_1^0)}\} \tag{74}$$

$$\Pi_{(3)}^{\alpha\beta i,\alpha'\beta' i^*} = -\frac{\mathrm{i}}{4}\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}_2-\boldsymbol{y}_1)}\frac{B_i^{\alpha\beta}(-\boldsymbol{p})A_{i^*}^{\alpha'\beta'}(\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+m_i^2}\left(\sqrt{\boldsymbol{p}^2+m_i^2}+\sqrt{\boldsymbol{p}^2+m_{i^*}^2}\right)} \times\{\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}(y_2^0-y_1^0)} + \mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_{i^*}^2}(y_2^0-y_1^0)}\} \tag{75}$$

$$\Pi_{(4)}^{\alpha\beta i,\alpha'\beta' i^*} = -\frac{\mathrm{i}}{4}\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}_2-\boldsymbol{y}_1)}\frac{A_i^{\alpha\beta}(-\boldsymbol{p})B_{i^*}^{\alpha'\beta'}(\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+m_{i^*}^2}\left(\sqrt{\boldsymbol{p}^2+m_i^2}+\sqrt{\boldsymbol{p}^2+m_{i^*}^2}\right)} \times\{\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}(y_2^0-y_1^0)} + \mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_{i^*}^2}(y_2^0-y_1^0)}\} \tag{76}$$

$$\Gamma_{(1)}^{\alpha\beta i,\alpha'\beta' i^*} = -\frac{1}{4}(y_2^0-y_1^0)\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}_2-\boldsymbol{y}_1)}A_i^{\alpha\beta}(-\boldsymbol{p}_1)A_{i^*}^{\alpha'\beta'}(\boldsymbol{p}_2) \times\int_0^1 \mathrm{d}s\,\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}(y_2^0-y_1^0)s}\,\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_{i^*}^2}(y_2^0-y_1^0)(1-s)} \tag{77}$$

$$\Gamma_{(2)}^{\alpha\beta i,\alpha'\beta' i^*} = \frac{1}{4}(y_2^0-y_1^0)\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}_2-\boldsymbol{y}_1)}\frac{B_i^{\alpha\beta}(-\boldsymbol{p})B_{i^*}^{\alpha'\beta'}(\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+m_i^2}\sqrt{\boldsymbol{p}^2+m_{i^*}^2}} \times\int_0^1 \mathrm{d}s\,\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}(y_2^0-y_1^0)s}\,\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_{i^*}^2}(y_2^0-y_1^0)(1-s)} \tag{78}$$

$$\Gamma_{(3)}^{\alpha\beta i,\alpha'\beta' i^*} = -\frac{\mathrm{i}}{4}(y_2^0-y_1^0)\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}_2-\boldsymbol{y}_1)}\frac{B_i^{\alpha\beta}(-\boldsymbol{p})A_{i^*}^{\alpha'\beta'}(\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+m_i^2}} \times\int_0^1 \mathrm{d}s\,\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}(y_2^0-y_1^0)s}\,\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_{i^*}^2}(y_2^0-y_1^0)(1-s)} \tag{79}$$

$$\Gamma_{(4)}^{\alpha\beta i,\alpha'\beta' i^*} = -\frac{\mathrm{i}}{4}(y_2^0-y_1^0)\int \mathrm{d}\boldsymbol{p}\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{y}_2-\boldsymbol{y}_1)}\frac{A_i^{\alpha\beta}(-\boldsymbol{p})B_{i^*}^{\alpha'\beta'}(\boldsymbol{p})}{\sqrt{\boldsymbol{p}^2+m_{i^*}^2}} \times\int_0^1 \mathrm{d}s\,\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_i^2}(y_2^0-y_1^0)s}\,\mathrm{e}^{-\sqrt{\boldsymbol{p}^2+m_{i^*}^2}(y_2^0-y_1^0)(1-s)}. \tag{80}$$

We obtain after some calculations that

$$S_2^{\alpha\beta,\alpha'\beta'}(y_2-y_1) = \int \mathrm{d}x\,\mathcal{D}_{\alpha\beta}^{-1}(x-y_1)\mathcal{D}_{\alpha'\beta'}^{-1}(x-y_2) \equiv \sum_{\delta=1}^{4}\left(\sum_{i,i^*}\Pi_\delta^{\alpha\beta i,\alpha'\beta' i^*} + \sum_{i,i^*}\Gamma_\delta^{\alpha\beta i,\alpha'\beta' i^*}\right). \tag{81}$$

From the explicit formulae (73)–(80) it follows that all the functions listed can be analytically continued in the variable $y_2^0 - y_1^0$ to the Minkowski space values $\mathrm{i}(y_2^0 - y_1^0)$

Proof of lemma 3.3. The following (see [4, equations 4.16])

$$\int \prod_{i=1}^{n} \mathrm{e}^{-\zeta_i |t-t_i|}\,\mathrm{d}t = \frac{1}{\zeta_1 + \cdots + \zeta_n}\mathrm{e}^{-\zeta_2(t_2-t_1)} \ldots \mathrm{e}^{-\zeta_n(t_n-t_1)} + \sum_{j=1}^{n-1}\prod_{i=1}^{j-1} \mathrm{e}^{-\zeta_i(t_j-t_i)}(t_{j+1}-t_j)$$
$$\times \prod_{i=j+2}^{n} \mathrm{e}^{-\zeta_i(t_i-t_{j+1})} \int_0^1 \mathrm{e}^{-[(\zeta_1+\cdots+\zeta_j)s + (\zeta_{j+1}+\cdots+\zeta_n)(1-s)(t_{j+1}-t_j)]}\,\mathrm{d}s$$
$$+ \frac{1}{\zeta_1 + \cdots + \zeta_n}\mathrm{e}^{-\zeta_1(t_n-t_1)} \ldots \mathrm{e}^{-\zeta_{n-1}(t_n-t_{n-1})} \tag{82}$$

is valid for any $t_1 < t_2 < \cdots < t_n$ and complex numbers ζ_i such that $\Re\zeta_i > 0$ for all i and the decomposition (66) is used to derive the following representation of $S_k^{;d}$:

$$S_k^{;d\alpha_1\beta_1\ldots\alpha_k\beta_k}(x_1, \ldots, x_k) \equiv \int \mathrm{d}x\,\mathcal{D}_A^{-1\alpha_1\beta_1}(x-x_1)\ldots\mathcal{D}_A^{-1\alpha_k\beta_k}(x-x_k)$$
$$= \sum_{\delta_1=1}^{n}\ldots\sum_{\delta_n=1}^{n}\int \mathrm{d}p_1^0\,\mathrm{d}\boldsymbol{p}_1\ldots\mathrm{d}p_k^0\,\mathrm{d}\boldsymbol{p}_k \prod_{j=1}^{k}\mathcal{W}_{\delta_k}^{\alpha_k\beta_k}(p_j^0, \boldsymbol{p}_j)$$
$$\times \int \mathrm{d}\boldsymbol{x} \prod_{j=1}^{k} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{x}_j)\cdot\boldsymbol{p}_j} \int \mathrm{d}x^0 \prod_{j=1}^{k} \mathrm{e}^{-p_j^0|x^0-x_j^0|}$$
$$= \sum_{\delta_1=1}^{n}\ldots\sum_{\delta_n=1}^{n}\int \mathrm{d}p_1^0\,\mathrm{d}\boldsymbol{p}_1\ldots\,\mathrm{d}p_k^0\,\mathrm{d}\boldsymbol{p}_k \int \mathrm{d}\boldsymbol{x}\prod_{j=1}^{k} \mathrm{e}^{-\mathrm{i}(\boldsymbol{x}-\boldsymbol{x}_j)\cdot\boldsymbol{p}_j}\prod_{j=1}^{k}\mathcal{W}_{\delta_k}^{\alpha_k\beta_k}(p_j^0, \boldsymbol{p}_j)$$
$$\times\left\{\frac{1}{p_1^0+\cdots+p_k^0}\prod_{j=1}^{k}\mathrm{e}^{-p_j^0(x_j^0-x_1^0)} + \sum_{j=1}^{n-1}\prod_{i=1}^{j-1}\mathrm{e}^{-p_i^0(x_j^0-x_i^0)}(x_{j+1}^0-x_j^0)\right.$$
$$\times\prod_{i=j+2}^{n}\mathrm{e}^{-p_i^0(x_i^0-x_{j+1}^0)}\int_0^1 \mathrm{d}s\,\mathrm{e}^{-[(p_1^0+\cdots+p_j^0)s+(p_{j+1}^0+\cdots+p_k^0)(1-s)(x_{j+1}^0-x_j^0)]}$$
$$\left.+\frac{1}{p_1^0+\cdots+p_k^0}\mathrm{e}^{-p_1^0(x_k^0-x_1^0)}\ldots\mathrm{e}^{-p_{k-1}^0(x_k^0-x_{k-1}^0)}\right\}. \tag{83}$$

Similarly, as in the proof of lemma 3.2, when using the explicit expressions for $\{\mathcal{W}_\delta^{\alpha\beta}(p_0, \boldsymbol{p})\}$ given in the proof of proposition 3.3 one can see that the functions $S_k^{\prime d\alpha_1\beta_1\ldots\alpha_k\beta_k}$ are given by sums, where each term is manifestly given by the Fourier–Laplace transforms of some tempered distribution supported on positive energies. □

Remark 3.4. Let $\{\mathcal{W}_n\}$ be a set of Wightman distributions, which are local, τ_M-covariant, and fulfil the weak form of the spectral axiom. Then, using a version of the GNS construction (see e.g. [46] for this), one can construct an inner product space $\mathcal{H}^{\mathrm{ph}}$ with the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}^{\mathrm{ph}}}$, a linear weakly continuous map

$$A_q : \mathcal{S}(\mathbb{R}^D)\otimes\mathbb{R}^N \longrightarrow \ell(\mathcal{H}^{\mathrm{ph}})$$

where $\ell(\mathcal{H}^{\mathrm{ph}})$ is the set of linear not necessarily bounded operators acting on $\mathcal{H}^{\mathrm{ph}}$, a nonunitary and unbounded representation $U_{\tau_M}^M$ of $\mathcal{P}_+^\uparrow(D)$ in $\mathcal{H}^{\mathrm{ph}}$ under which the quantum-field operator A_q transforms covariantly, and a vector Ω, the physical vacuum, which is cyclic with respect to the action of $A_q(f)$ and invariant with respect to $U_{\tau_M}^M$.

4. Examples in $D = 3$

The complete description of the set of all covariant operators $\mathcal{D} \in \mathrm{Cov}(\tau)$ where τ is any finite dimensional representation of the group $\boldsymbol{SO}(\mathbf{3})$ or $\boldsymbol{SO}(\mathbf{1},\mathbf{3})$ is given in the monographs [17, 35], see also [32]. To illustrate our general theory developed in the previous paragraphs we focus on the lowest-dimensional real representations $D_0 \oplus D_1$, $D_1 \oplus D_1$ and $D_{\frac{1}{2}} \oplus D_{\frac{1}{2}}$ of the group $\boldsymbol{SO}(\mathbf{3})$. The much more interesting case $D = 4$ shall be analysed in a greater details in our forthcoming paper [19]. The examples presented below do not cover all the possibilities. The crucial point is that we have used a rather special realification procedure in order to transform the complex description of the sets $\mathrm{Cov}(\tau)$ given in [17, 35, 32] into a manifestly real form. Our realification is achieved by a certain similarity transformation, fixed by the choice of a realification matrix E_τ. Different choices of the realification procedure may lead to different families of covariant operators, not necessary connected by a real similarity transformation.

4.1. $D_0 \oplus D_1$: Higgs-like models

This class of models described a doublet of fields $\varphi = (\varphi_0, \boldsymbol{A})$ where φ_0 is the scalar field and $\boldsymbol{A}$ is the vector field, coupled by noise through the corresponding covariant SPDE of the form (33).

The realification matrix $E_{(0,1)}$ is chosen to be,

$$E_{(0,1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & -\mathrm{i} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \mathrm{i}\sqrt{2} & 0 \end{pmatrix}. \tag{84}$$

The real form of the corresponding covariant operators $\mathcal{D}_{(0,1)} \in \mathrm{Cov}((D_0 \oplus D_1)^R)$ with respect to $SO(3)$ with the mass term $M = m_0\mathbf{1}_0 \oplus m_1\mathbf{1}_3$ is given by

$$\hat{\mathcal{D}}_{(0,1)}(p) = \begin{pmatrix} m_0 & a\mathrm{i}p_0 & a\mathrm{i}p_1 & a\mathrm{i}p_2 \\ b\mathrm{i}p_0 & m_1 & -c\mathrm{i}p_2 & c\mathrm{i}p_1 \\ b\mathrm{i}p_1 & c\mathrm{i}p_2 & m_1 & -c\mathrm{i}p_0 \\ b\mathrm{i}p_2 & -c\mathrm{i}p_1 & c\mathrm{i}p_0 & m_1 \end{pmatrix} \tag{85}$$

with $a, b, c \in \mathbb{R}$ with $\det \hat{\mathcal{D}}_{(0,1)}(ip) = (-c^2p^2 + m_1^2)(abp^2 + m_0m_1)$

To obtain an admissible mass spectrum we need to put either $c = 0$ or $m_1 = 0$, thus giving up the ellipticity of $\hat{\mathcal{D}}$. Admissible covariant operators are obtained iff $c = 0$ if $m_1 \neq 0$ or $m_1 = 0$. We add that the operator is covariant with respect to $O(3)$ iff $c = 0$. The Green function is given by

$$\hat{\mathcal{D}}^{-1}_{(0,1)}(\mathrm{i}p) = \frac{1}{abp^2 + m_0m_1} \left(\begin{array}{c|ccc} m_1 & -a\mathrm{i}p_0 & -a\mathrm{i}p_1 & -a\mathrm{i}p_2 \\ \hline -b\mathrm{i}p_0 & & & \\ -b\mathrm{i}p_1 & & G_{\mu\nu}(p) & \\ -b\mathrm{i}p_2 & & & \end{array}\right) \tag{86}$$

where

$$G_{\mu\nu}(p) = \frac{1}{-c^2p^2 + m_1^2}\{(abp^2 + m_0m_1)(m_1\delta_{\mu\nu} + c\mathrm{i}\varepsilon_{\mu\nu\lambda}p_\lambda) - p_\mu p_\nu(abm_1 + c^2m_0)\}$$

for $\mu, \nu \in \{0, 1, 2\}$. The corresponding two-point function (more precisely the contribution coming from the Poisson piece of noise and not integrated with Lévy measure ν, see equation 2.50):

$$\hat{S}^{(2)}_{(0,1)}(p,\alpha) = (\hat{S}^{(2)}_{kl}(p,\alpha)) = \left(\begin{array}{c|ccc} \hat{S}^{(2)}_{33}(p,\alpha) & \hat{S}^{(2)}_{30}(p,\alpha) & \hat{S}^{(2)}_{31}(p,\alpha) & \hat{S}^{(2)}_{32}(p,\alpha) \\ \hline \begin{array}{c}\hat{S}^{(2)}_{03}(p,\alpha)\\ \hat{S}^{(2)}_{13}(p,\alpha)\\ \hat{S}^{(2)}_{23}(p,\alpha)\end{array} & & \hat{S}^{(2)}_{\mu\nu}(p,\alpha) & \end{array}\right) \tag{87}$$

where

$$\hat{S}^{(2)}_{33}(p,\alpha) = |m_1\alpha_3 - \mathrm{i}b\alpha_\mu p_\mu|^2/(abp^2 + m_0 m_1)^2$$

$$\begin{aligned}\hat{S}^{(2)}_{3\mu}(p,\alpha) = \hat{S}^{(2)}_{\mu 3}(-p,\alpha) = {} & (m_1\alpha_3 - \mathrm{i}b\alpha_\lambda p_\lambda)[m_1(abp^2 + m_0 m_1)\alpha_\mu \\ & +\mathrm{i}a(-c^2p^2 + m_1^2)\alpha_3 p_\mu - (abm_1 + c^2 m_0)\alpha_\lambda p_\lambda p_\mu] \\ & /(-c^2p^2 + m_1^2)(abp^2 + m_0 m_1)^2\end{aligned}$$

$$\begin{aligned}\hat{S}^{(2)}_{\mu\nu}(p,\alpha) = {} & [a^2\alpha_3^2 + (\alpha_\lambda p_\lambda)^2(abm_1 + c^2m_0)^2/(-c^2p^2 + m_1^2)^2]p_\mu p_\nu/(abp^2 + m_0m_1)^2 \\ & +m_1^2\alpha_\mu\alpha_\nu/(-c^2p^2 + m_1^2)^2 - m_1(abm_1 + c^2m_0)\alpha_\lambda p_\lambda(p_\mu\alpha_\nu + p_\nu\alpha_\mu) \\ & /(-c^2p^2 + m_1^2)^2(abp^2 + m_0m_1) \\ & -\mathrm{i}am_1\alpha_3(p_\mu\alpha_\nu - p_\nu\alpha_\mu)/(-c^2p^2 + m_1^2)(abp^2 + m_0m_1)\end{aligned}$$

for $\mu, \nu \in \{0, 1, 2\}$.

Above we have used the notation for the variable of Lévy measure: $\alpha \equiv (\alpha_3, \alpha_0, \alpha_1, \alpha_2)$.

Remarks. The representation $D_0 \oplus D_1$ is also of quaternionic type [13]. Choosing $m_0^2 + m_1^2 = 0$ and $a = -1, b = 1, c = 1$ (respectively $a = -1, b = 1, c = -1$) we obtain the purely quaternionic description of the corresponding Clifford algebra of $\mathbb{R}^3$ Dirac operators. More explicitly let $C(\mathbb{R}^3)$ denote the corresponding Clifford algebra over $\mathbb{R}^3$ and $\Lambda(\mathbb{R}^3)$ the exterior algebra of $\mathbb{R}^3$. Let us denote by $C(\mathbb{R}^3) = C_+(\mathbb{R}^3) \oplus C_-(\mathbb{R}^3)$ (respectively by $\Lambda(\mathbb{R}^3) = \Lambda_+(\mathbb{R}^3) \oplus \Lambda_-(\mathbb{R}^3)$) the canonical decomposition of $C(\mathbb{R}^3)$ (respectively of $\Lambda(\mathbb{R}^3)$). Let $\mathbb{H}$ stand for the noncommutative field of quaternions with the base $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Noting that $C(\mathbb{R}^3) \cong \mathbb{H} \oplus \mathbb{H}$ and $\Lambda(\mathbb{R}^3) \cong \mathbb{H} \oplus \mathbb{H}$ and using two nonequivalent representations of $\mathbb{H}$ on $\mathbb{H}$ given by left (respectively right) multiplication we obtain the following explicit expressions for the corresponding left (respectively right) Dirac operator of $\Lambda(\mathbb{R}^3)$:

$$\mathcal{D}_{\mathrm{L}} \equiv L(\boldsymbol{i})\partial_0 + L(\boldsymbol{j})\partial_1 + L(\boldsymbol{k})\partial_2 \equiv \begin{pmatrix} 0 & -\partial_0 & -\partial_1 & -\partial_2 \\ \partial_0 & 0 & -\partial_2 & \partial_1 \\ \partial_1 & \partial_2 & 0 & -\partial_0 \\ \partial_2 & -\partial_1 & \partial_0 & 0 \end{pmatrix} \tag{88}$$

respectively

$$\mathcal{D}_{\mathrm{R}} \equiv R(\boldsymbol{i})\partial_0 + R(\boldsymbol{j})\partial_1 + R(\boldsymbol{k})\partial_2 \equiv \begin{pmatrix} 0 & -\partial_0 & -\partial_1 & -\partial_2 \\ \partial_0 & 0 & \partial_2 & -\partial_1 \\ \partial_1 & -\partial_2 & 0 & \partial_0 \\ \partial_2 & \partial_1 & -\partial_0 & 0 \end{pmatrix} \tag{89}$$

with the properties

$$\mathcal{D}_{\mathrm{L}}\mathcal{D}_{\mathrm{L}}^* = -\triangle_3 \mathbf{1}_4$$

where $\mathcal{D}^* = -\mathcal{D}^{\mathrm{T}}$, and respectively

$$\mathcal{D}_{\mathrm{R}}\mathcal{D}_{\mathrm{R}}^* = -\triangle_3 \mathbf{1}_4$$

where $\mathcal{D}_{\mathrm{L}}^* = -L(\boldsymbol{i})\partial_0 - L(\boldsymbol{j})\partial_1 - L(\boldsymbol{k})\partial_2$ (respectively $\mathcal{D}_{\mathrm{R}}^* = -R(\boldsymbol{i})\partial_0 - R(\boldsymbol{j})\partial_1 - R(\boldsymbol{k})\partial_2$). Another simple covariant decomposition of the three-dimensional Laplacian $-\triangle_3$ can be described by

$$\mathcal{D} = \begin{pmatrix} 0 & \partial_0 & \partial_1 & \partial_2 \\ \partial_0 & 0 & -\partial_2 & \partial_1 \\ \partial_1 & \partial_2 & 0 & -\partial_0 \\ \partial_2 & -\partial_1 & \partial_0 & 0 \end{pmatrix} \tag{90}$$

and

$$\mathcal{D}^{\mathrm{T}} = \begin{pmatrix} 0 & \partial_0 & \partial_1 & \partial_2 \\ \partial_0 & 0 & \partial_2 & -\partial_1 \\ \partial_1 & -\partial_2 & 0 & \partial_0 \\ \partial_2 & \partial_1 & -\partial_0 & 0 \end{pmatrix} \tag{91}$$

and then $\mathcal{D}\mathcal{D}^{\mathrm{T}} = \triangle_3 \mathbf{1}_4$. This corresponds to the choice $a = 1, b = 1$ and $c = +1$ (respectively -1) in (85). The question of the covariance properties of this decomposition was the starting point of this research.

4.2. $D_1 \oplus D_1$: *Interacting vector fields*

The models of this sort describe a doublet of vector fields $\boldsymbol{A} = (A_0, A_1, A_2)$, $\boldsymbol{B} = (B_0, B_1, B_2)$ coupled to itself by the noise in the corresponding covariant SPDE.

The realification matrix $E_{(1,1)}$ is chosen to be:

$$E_{(1,1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} & 0 & -\mathrm{i} & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & \mathrm{i}\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} & 0 & -\mathrm{i} \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \mathrm{i}\sqrt{2} & 0 \end{pmatrix}. \tag{92}$$

The manifestly real expressions for $\mathcal{D}_{(1,1)} \in \mathrm{Cov}((D_1 \oplus D_1)^R)$ obtained by the application of $E_{(1,1)}$ are given by:

$$\hat{\mathcal{D}}_{(1,1)}(p) = \begin{pmatrix} m_1 & -a\mathrm{i}p_2 & a\mathrm{i}p_1 & 0 & -b\mathrm{i}p_2 & b\mathrm{i}p_1 \\ a\mathrm{i}p_2 & m_1 & -a\mathrm{i}p_0 & b\mathrm{i}p_2 & 0 & -b\mathrm{i}p_0 \\ -a\mathrm{i}p_1 & a\mathrm{i}p_0 & m_1 & -b\mathrm{i}p_1 & b\mathrm{i}p_0 & 0 \\ 0 & -c\mathrm{i}p_2 & c\mathrm{i}p_1 & m_2 & -d\mathrm{i}p_2 & d\mathrm{i}p_1 \\ c\mathrm{i}p_2 & 0 & -c\mathrm{i}p_0 & d\mathrm{i}p_2 & m_2 & -d\mathrm{i}p_0 \\ -c\mathrm{i}p_1 & c\mathrm{i}p_0 & 0 & -d\mathrm{i}p_1 & d\mathrm{i}p_0 & m_2 \end{pmatrix} \tag{93}$$

where the central element $M = m_1\mathbf{1}_3 \oplus m_2\mathbf{1}_3$, $m_1, m_2 \in \mathbb{R}$.

Computing $\det \hat{\mathcal{D}}_{(1,1)}(\mathrm{i}p)$ we obtain:

$$\det \hat{\mathcal{D}}_{(1,1)}(\mathrm{i}p) = m_1 m_2 \{(ad - bc)^2 p^4 + ((m_2^2 a^2) + 2bcm_1m_2 + m_1^2 d^2) p^2 + m_1^2 m_2^2\}.$$

The conditions for the proper mass spectrum could be easily obtained as $\hat{\mathcal{D}}_{(1,1)}(\mathrm{i}p)$ is a biquadratic polynom. Provided that $\det\hat{\mathcal{D}}_{(1,1)}(\mathrm{i}p) \neq 0$ we can invert the matrix $\hat{\mathcal{D}}_{(1,1)}(\mathrm{i}p)$ obtaining the corresponding Green function

$$\hat{\mathcal{D}}^{-1}_{(1,1)}(p,\alpha) = \frac{1}{\{f^2p^4 - (h^2 - 2fm_1m_2)p^2 + m_1^2m_2^2\}} \left(\begin{array}{c|c} G^{(1,1)}_{\mu\nu}(p,\alpha) & G^{(1,2)}_{\mu\nu}(p,\alpha) \\ \hline G^{(2,1)}_{\mu\nu}(p,\alpha) & G^{(2,2)}_{\mu\nu}(p,\alpha) \end{array}\right) \tag{94}$$

where we have $f \equiv ad - bc$, $h \equiv am_2 + dm_1$ and

$$\begin{aligned} G^{(1,1)}_{\mu\nu}(p,\alpha) &= m_1m_2(-\mathrm{e}_1p^2 + m_1m_2^2)\delta_{\mu\nu} + m_2(f^2p^2 - m_2e_2)p_\mu p_\nu \\ &\quad + m_1m_2(-dfp^2 + am_2^2)\mathrm{i}\varepsilon_{\mu\nu\lambda}p_\lambda \\ G^{(1,2)}_{\mu\nu}(p,\alpha) &= bm_1m_2\{hp^2\delta_{\mu\nu} - hp_\mu p_\nu + (fp^2 + m_1m_2)\mathrm{i}\varepsilon_{\mu\nu\lambda}p_\lambda\} \end{aligned}$$

for $\mu,\nu \in \{0,1,2\}$ with $e_1 \equiv (d^2m_1 + bcm_2)$, $e_2 \equiv (a^2m_2 + bcm_1)$.

The two last blocks of the Green matrix can be obtained by making the following exchanges: $a \leftrightarrow d$, $m_1 \leftrightarrow m_2$ within the $G^{(1,1)}_{\mu\nu}(p,\alpha)$ matrix to get $G^{(2,2)}_{\mu\nu}(p,\alpha)$ and $b \leftrightarrow c$ within $G^{(1,2)}_{\mu\nu}(p,\alpha)$ to get $G^{(2,1)}_{\mu\nu}(p,\alpha)$. The corresponding two-point Schwinger function (more precisely the contribution coming from Poisson piece of the noise without integration over ν as in the previous case) is given by

$$\hat{S}^{(2)}_{(1,1)}(p,\alpha) = \frac{1}{\{f^2p^4 - (h^2 - 2fm_1m_2)p^2 + m_1^2m_2^2\}^2} \left(\begin{array}{c|c} \hat{S}^{(1,1)}_{\mu\nu}(p,\alpha) & \hat{S}^{(1,2)}_{\mu\nu}(p,\alpha) \\ \hline \hat{S}^{(2,1)}_{\mu\nu}(p,\alpha) & \hat{S}^{(2,2)}_{\mu\nu}(p,\alpha) \end{array}\right) \tag{95}$$

with

$$\begin{aligned} \hat{S}^{(1,1)}_{\mu\nu}(p,\alpha) &= [m_2(f^2p^2 - m_2e_2)\alpha_\lambda p_\lambda - cm_1m_2h\beta_\lambda p_\lambda]^2 p_\mu p_\nu \\ &\quad + [m_2(f^2p^2 - m_2e_2)\alpha_\lambda p_\lambda - cm_1m_2h\beta_\lambda p_\lambda] \\ &\quad \times [m_1m_2(-e_1p^2 + m_1m_2^2)(p_\mu\alpha_\nu + p_\nu\alpha_\mu) + cm_1m_2hp^2(p_\mu\beta_\nu + p_\nu\beta_\mu)] \\ &\quad + cm_1^2m_2^2hp^2(-e_1p^2 + m_1m_2^2)(\alpha_\mu\beta_\nu + \alpha_\nu\beta_\mu) \\ &\quad + m_1^2m_2^2(-e_1p^2 + m_1m_2^2)^2\alpha_\mu\alpha_\nu + (cm_1m_2hp^2)^2\beta_\mu\beta_\nu \\ \hat{S}^{(1,2)}_{\mu\nu}(p,\alpha) &= -m_1m_2\{hbm_2(f^2p^2 - m_2e_2)(\alpha_\lambda p_\lambda)^2 \\ &\quad + hcm_1(f^2p^2 - e_1m_1)(\beta_\lambda p_\lambda)^2 - \alpha_\lambda p_\lambda\beta_\omega p_\omega \\ &\quad \times [(f^2p^2 - m_2e_2)(f^2p^2 - m_1e_1) + m_1m_2bch^2]\}p_\mu p_\nu + m_1m_2^2 \\ &\quad \times bhp^2[(f^2p^2 - m_2e_2)\alpha_\lambda p_\lambda - m_1ch\beta_\lambda p_\lambda]p_\mu\alpha_\nu + m_1^2m_2(-e_1p^2 + m_1m_2^2) \\ &\quad \times [(f^2p^2 - m_1e_1)\beta_\lambda p_\lambda - m_2bh\alpha_\lambda p_\lambda]p_\nu\alpha_\mu + m_1m_2^2(-e_2p^2 + m_1^2m_2) \\ &\quad \times [(f^2p^2 - m_2e_2)\alpha_\lambda p_\lambda - m_1ch\beta_\lambda p_\lambda]p_\mu\beta_\nu + m_1^2m_2chp^2 \\ &\quad \times [(f^2p^2 - m_1e_1)\beta_\lambda p_\lambda - bm_2h\alpha_\lambda p_\lambda]p_\nu\beta_\mu + m_1^2m_2^2hp^2 \\ &\quad \times [b(-e_1p^2 + m_1m_2^2)\alpha_\mu\alpha_\nu + c(-e_2p^2 + m_1^2m_2)\beta_\mu\beta_\nu] \\ &\quad + m_1^2m_2^2(-e_1p^2 + m_1m_2^2)(-e_2p^2 + m_1^2m_2)\alpha_\mu\beta_\nu + bc(m_1m_2hp^2)^2\alpha_\nu\beta_\mu \end{aligned}$$

for $\mu,\nu \in \{0,1,2\}$.

We use the notation for the variable of Lévy measure: $\alpha \equiv (\alpha_0,\alpha_1,\alpha_2,\beta_0,\beta_1,\beta_2)$. The block $\hat{S}^{(2,2)}_{\mu\nu}(p,\alpha)$ can be obtained by the exchanges $a \leftrightarrow d$, $m_1 \leftrightarrow m_2$ and $b \leftrightarrow c$ in the block $\hat{S}^{(1,1)}_{\mu\nu}(p,\alpha)$ and the block $\hat{S}^{(2,1)}_{\mu\nu}(p,\alpha)$ by $b \leftrightarrow c$ within the block $\hat{S}^{(1,2)}_{\mu\nu}(p,\alpha)$.

It is worthwhile to observe that in the variety of covariant operators there does not exists a reflection covariant operator. By specialization of the parameters of the covariant operator we can find in the Gaussian part of two-point Schwinger function the Euclidean two-point function of two copies of the so-called Euclidean Proca field introduced in [22, 23, 52, 53]. If we put

$$a = d = 0 \qquad b^2 = c^2 = 1 \qquad bc = -1 \qquad \text{and} \qquad m_1 = m_2 = m \tag{96}$$

then we obtain for the Gaussian part

$$\hat{S}^{(2)}_{\mathrm{G};(1,1))}(p) = \left(\begin{array}{c|c} \left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)\frac{1}{p^2+m^2} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \left(\delta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)\frac{1}{p^2+m^2} \end{array}\right). \tag{97}$$

The corresponding covariance matrix is $A = \mathbf{1}_6$ (see equation 2.17).

4.3. *The* $D_{\frac{1}{2}} \oplus D_{\frac{1}{2}}$*-case*

This representation seems not to be of physical interest since it contradicts usual spin-statistic connection. We note that in the case of nonpositive quantum-field theory the usual connection between spin and statistic could be violated [12]. We use the $D_{\frac{1}{2}} \oplus D_{\frac{1}{2}}$-covariant noise η. In this context the study of realifications of this representation could be more useful than the analysis of the corresponding covariant operators, Green and Schwinger functions. However, we mention this case to complete the list of the lowest dimensional cases.

The realification matrix $E_{(\frac{1}{2},\frac{1}{2})}$ is

$$E_{(\frac{1}{2},\frac{1}{2})} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ \mathrm{i} & 0 & 0 & -\mathrm{i} \\ 0 & 1 & -1 & 0 \\ 0 & \mathrm{i} & \mathrm{i} & 0 \end{pmatrix}. \tag{98}$$

The covariant operator in the Fourier representation is

$$\hat{\mathcal{D}}_{(\frac{1}{2},\frac{1}{2})}(p) = \left(\begin{matrix} c\mathrm{i}p_0 - d\mathrm{i}p_1 - a\mathrm{i}p_2 + m & -d\mathrm{i}p_0 - c\mathrm{i}p_1 - b\mathrm{i}p_2 \\ -d\mathrm{i}p_0 - c\mathrm{i}p_1 + b\mathrm{i}p_2 & -c\mathrm{i}p_0 + d\mathrm{i}p_1 - a\mathrm{i}p_2 + m \\ a\mathrm{i}p_0 + b\mathrm{i}p_1 + c\mathrm{i}p_2 & b\mathrm{i}p_0 - a\mathrm{i}p_1 - d\mathrm{i}p_2 \\ -b\mathrm{i}p_0 + a\mathrm{i}p_1 - d\mathrm{i}p_2 & a\mathrm{i}p_0 + b\mathrm{i}p_1 - c\mathrm{i}p_2 \end{matrix}\right.$$
$$\left.\begin{matrix} a\mathrm{i}p_0 - b\mathrm{i}p_1 + c\mathrm{i}p_2 & b\mathrm{i}p_0 + a\mathrm{i}p_1 - d\mathrm{i}p_2 \\ -b\mathrm{i}p_0 - a\mathrm{i}p_1 - d\mathrm{i}p_2 & a\mathrm{i}p_0 - b\mathrm{i}p_1 - c\mathrm{i}p_2 \\ -c\mathrm{i}p_0 - d\mathrm{i}p_1 + a\mathrm{i}p_2 + m & d\mathrm{i}p_0 - c\mathrm{i}p_1 + b\mathrm{i}p_2 \\ d\mathrm{i}p_0 - c\mathrm{i}p_1 - b\mathrm{i}p_2 & c\mathrm{i}p_0 + d\mathrm{i}p_1 + a\mathrm{i}p_2 + m \end{matrix}\right) \tag{99}$$

with $a, b, c, d \in \mathbb{R}$ and $\det(\hat{\mathcal{D}}_{(\frac{1}{2},\frac{1}{2})}(p)) = [(a^2 + b^2 + c^2 + d^2)p^2 + m^2]^2 - 4m^2b^2p^2$.

We can get the admissible mass spectrum taking, for example, $b = 0$.

We can use the methods presented above to obtain explicit formulae for the Green functions and the Schwinger functions. The corresponding expressions are much more complicated than in the examples above and will therefore not be presented here.

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